M.S c Mathematics –SEM 3 Functional Analysis-L-7 CC-11 Unit 1

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Example(1)

Construct an example of a normed linear space which is not a Banach space

Solution:

The real linear space E of polynomials over [0,1] with real coefficient and the norm defined by

$||P|| = \sup\limits_{0 \le t \le 1} |P(t)| \quad for \;\; P \in E$ is a real normed space which is not a Banach space.

Proof: Clearly $||P|| \geq 0, ||P|| = 0, iff \; P = 0$

For any real a , $||aP|| = \sup\limits_{0 \le t \le 1} |P(t)|$

$$= |a| \sup_{0 \le t \le 1} |P(t)|$$

$$= |a|. ||P||$$

Also, $|P(t) + q(t)| \leq |P(t)| + |q(t)|$

$$\leq ||P|| + ||q||$$

Hence $||P + q|| = \operatorname*{Sup}_{0 \leq t \leq 1} |P(t) + q(t)|$

$$\leq ||P|| + ||q||$$

Hence E is a real normed linear space .The metric d defined by the norm is given by

$$d(p, q) = \operatorname*{Sup}_{0 \leq t \leq 1} |P(t) - q(t)|$$

$$for\ all\ p, q \in E$$

But (E, d) is not a metric space.

Since it is not a Banach space

For this,

We consider the sequence (P_n) of polynomials defined by

$$P_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^n}{n!}$$

$$If\ n > m$$

$$then\ d(P_n, P_m) = ||P_n - P_m||$$

$$= \sup_{0 \le t \le 1} |P_n(t) - P_m(t)|$$

$$= \sup_{0 \le t \le 1} \left| \frac{t^{m+1}}{m+1!} + + \frac{t^n}{n!} \right|$$

$$\le \frac{1}{m+1!} + . \text{.................} \frac{1}{n!} \text{......(1)}$$

Let $\in > 0$ be arbitrary

Then from quation (1) it can be made less than $\in$ by choosing m sufficiently large.

So ,P_n is a Cauchy sequence

But this sequence does not converge to any element of E.

So ,infact in C[0,1] with sup metric ,$(P_n(t))$ converges to e^t which is not a polynomial.

Thus E is an incomplete normed linear space .

i.e a normed linear space is not a Banach space

Example(2)

Construct a metric linear space which is not a normed linear space

A linear space E with a metric ρ need not be normed linear space.

In the sense that it is not always possible to define a norm on E which will generate the given metric ρ.

To see this consider the metric space S of all numerical sequences.

S is a linear space.

If we define for $x = (x_i), y = (y_i)$ for S

$$x + y = (x_i + y_i)$$

$ax = (ax_i)$, **a being scalar.**

So, we consider the metric space ρ on S defined by

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i}{1 + |x_i - y_i|}$$

It is not possible to define a norm on S which will generate the metric ρ.

For let any norm $||\ .||$, s be given and let d be the metric generated by the norm.

It is sufficient to show that d is not equivalent to ρ.

For this we consider

$$e^{(i)} = (0, 0, \ldots \ldots \ldots \ldots, 1, 0, 0 \ldots \ldots \ldots) \in S$$

where we have 1 at the ith place & 0 everywhere else.

So we define

$$x^i = \frac{e^{(i)}}{||e^{(i)}||}, i = 1, 2, 3, \ldots \ldots \ldots$$

Now $\rho(x^{(i)}, 0) \leq \frac{1}{2^i} \to 0 \; as \; i \to \infty$

Hence $(x^{(i)}) \to 0$ in the sense of the metric ρ.

But $d(x^{(i)}, 0) = ||x^{(i)} - 0||$

$$= ||x^i|| = 1$$

Hence $(x^{(i)})$ does not converges to 0 with respect in the metric d.

Hence d is not equivalent to ρ then E is a metric linear space but not a normed linear space.