e-content (lecture-5)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic: Theorem and problem based on the Hilbert space .

Theorem: (The polarization identity)

If x and y are any two elements in a complex inner product space E(or a complex Hilbert space E) Then

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

Proof: *we have*

$$\|x + y\|^2 = (x + y, x + y)$$

$$= (x, x) + (y, x) + (x, y) + (y, y)$$

And $\|x - y\|^2 = (x - y, x - y)$

$$= (x, x) - (y, x) - (x, y) + (y, y)$$

Therefore

$$\|x+y\|^2 - \|x-y\|^2 = 2(y,x) + 2(x,y) \dots (1)$$

Replacing y *by* iy *in (1) we get*

$$\|x+iy\|^2 - \|x-iy\|^2 = 2(iy,x) + 2(x,iy)$$

$$= 2i(y,x) + 2\bar{\imath}(x,y)$$

$$= 2i(y,x) - 2i(x,y) \dots\dots (2)$$

Now multiplying both sides of (2) i *we get*

$$i\|x+iy\|^2 - i\|x-iy\|^2 = -2(y,x) + 2(x,y) \dots (3)$$

Adding (1) and (3) we get

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2$$
$$+ i\|x+iy\|^2 - i\|x-iy\|^2.$$

Problem: *Construct a Banach space of continuous functions which is not a Hilbert space*.

Solution: *we consider the Banach space* C[0,1] *of all continuous functions on the closed interval* [0,1] *of R with the norm defined by*

$$\|f\| = Sup\{|f(t)| : t \in [0,1]\} \quad \text{for } f \in C[0,1],$$

We show that this norm does not satisfy the parallelogram law.

Let $f(t) = t$ *and* $g(t) = 1 - t$ *be two functions from*[0,1] to *R. Then* $f(t) = t$ *and* $g(t) = 1 - t$ *are continuous functions on* [0,1].hence $f, g \in C[0,1]$.

Now by definition of norm we get

$$\|f\| = Sup\{|f(t)|: t \in [0,1]\}$$

$$= Sup\{|t|: t \in [0,1]\}$$

$$= Sup\{t : t \in [0,1]\}$$

$$= Sup[0,1] = 1$$

Again, $$\|g\| = Sup\{|g(t)|: t \in [0,1]\}$$

$$= Sup\{|1 - t|: t \in [0,1]\}$$

$$= Sup\{1 - t : t \in [0,1]\}$$

$$= 1$$

Now $$\|f + g\| = Sup\{|(f + g)(t)|: t \in [0,1]\}$$
$$= Sup\{|f(t) + g(t)|: t \in [0,1]\}$$

$$= Sup\{|t + 1 - t| : t \in [0,1]\}$$
$$= Sup\{1 : t \in [0,1]\} = 1$$

$$\|f - g\| = Sup\{|(f - g)(t)| : t \in [0,1]\}$$
$$= Sup\{|f(t) - g(t)| : t \in [0,1]\}$$
$$= Sup\{|t - 1 + t| : t \in [0,1]\}$$
$$= Sup\{|2t - 1| : t \in [0,1]\} = 1.$$

Thus $\|f + g\|^2 + \|f - g\|^2 = 1^2 + 1^2 = 2$

But $\quad 2\|f\|^2 + 2\|g\|^2 = 2.1^2 + 2.1^2 = 4$

Therefore $\|f + g\|^2 + \|f - g\|^2 \neq 2\|f\|^2 + 2\|g\|^2$.

Hence the parallelogram law is not satisfied for the suprimum norm on C[0,1]. Hence C[0,1] is *not a Hilbert space but it is a Banach space .*

END.