e-content

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SEM-3 CC-11 Unit-4(Functional Analysis)

Topic: Theorems based on the Hilbert Spaces.

Theorem:

(Cauchy-Schwarz Inequality or Schwarz Inequality)

Let E be an inner product space or a Hilbert space.

Then for all $x, y \in E$, $|(x, y)| \leq \|x\|.\|y\|$

Equality holds iff x and y are linearly dependent.

Proof: *we have for any scalar a,*

$$0 \leq (x - ay, x - ay)$$

$$= (x, x) - \bar{a}(x, y) - a(y, x) + a\bar{a}(y, y)$$

$$= \|x\|^2 - \bar{a}(x, y) - a(y, x) + |a|^2\|y\|^2 \ldots. \quad (1)$$

If $(y, x) = 0$ *then* $|(x, y)| = |\overline{(y, x)}| = |\bar{0}| = |0| = 0$

and 0$\leq \|x\|.\|y\|$ so $|(x,y)| \leq \|x\|.\|y\|$.

If $(y,x) \neq 0$ *then putting* $a = \frac{\|x\|^2}{(y,x)}$ *in (1) we get*

$$0 \leq \|x\|^2 - \|x\|^2 - \|x\|^2 + \frac{\|x\|^4}{|(y,x)|^2}\|y\|^2$$

Hence $\quad |(y,x)|^2 \leq \|x\|^2.\|y\|^2$

$$\Rightarrow |(x,y)| \leq \|x\|.\|y\|$$

$Now\ |(x,y)| = \|x\|.\|y\| \Leftrightarrow 0 = (x - ay, x - ay)$

$$\Leftrightarrow 0 = x - ay \Leftrightarrow x = ay$$

$$\Leftrightarrow x \text{ and } y \text{ are linearly dependent.}$$

Theorem:

(Continuity of inner product function in a Hilbert space)

Let $x_n \rightarrow x$ *and* $y_n \rightarrow y$ *in an inner product space E or in a Hilbert space E. Then* $(x_n, y_n) \rightarrow (x,y) i.e$ *the inner product function is jointly continuous in an inner product*

Space or in a Hilbert space .

Proof: *Let* $x_n \rightarrow x$ *and* $y_n \rightarrow y$ *in an inner product space E.*

Then $\quad |(x_n, y_n) - (x, y)|$

$$= |(x_n, y_n) - (x, y_n) + (x, y_n) - (x, y)|$$

$$= |(x_n - x, y_n) + (x, y_n - y)|$$

$$\leq |(x_n - x, y_n)| + |(x, y_n - y)|$$

$$\leq \|x_n - x\|\|y_n\| + \|x_n\|\|y_n - y\| \ldots(1)$$

Since $x_n \to x$ *and* $y_n \to y$ *so we have*

$$\|x_n - x\| \to 0 \text{ and } \|y_n - y\| \to 0$$

So $\|x_n - x\|\|y_n\| + \|x_n\|\|y_n - y\| \to 0$ as $n \to \infty$

Hence $|(x_n, y_n) - (x, y)| \to 0$ as $n \to \infty$

Therefore $(x_n, y_n) \to (x, y)$.

So the inner product function is a continuous function.

Theorem (Parallelogram law): *For any two elements* x *and* y *in an inner product space E or in a Hilbert space E we have*

$$\|x+y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\|y\|^2$$

Proof: *since* $\|x+y\|^2 = (x + y, x + y)$

$$= (x, x) + (x, y) + (y, x) + (y, y)$$

$$= \|x\|^2 + (x, y) + (y, x) + \|y\|^2 \text{....(1)}$$

Again $\|x-y\|^2 = (x - y, x - y)$

$$= (x, x) - (x, y) - (y, x) + (y, y)$$

$$= \|x\|^2 - (x, y) - (y, x) + \|y\|^2 \ldots (2)$$

Adding (1) and (2) we get

$$\|x+y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\|y\|^2.$$

End .