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sem-3 CC-11(Unit-4)

Topic: Problems based on inner product space.

Problem: Construct an inner product space which is not a Hilbert space or construct an incomplete inner product space.

Solution: we show that the linear space P[0,1]*of all real-valued polynomials on* [0,1] *with inner product given by* $(f,g) = \int_0^1 f(t)g(t)dt \ \ for \ \ f,g$ ∈ P[0,1],*is an inner product space which is not a Hilbert space .*

Clearly $(f,f) = \int_0^1 [f(t)]^2 \, dt \geq 0$

$$(f,f) = 0$$

$$iff \int_0^1 [f(t)]^2 \, dt = 0$$

$$iff \quad f(t) = 0 \;\; \forall \; t \in [0,1]$$

Iff $\quad f = 0.$

$$(g, f) = \int_0^1 g(t) f(t) dt = \int_0^1 f(t) g(t) dt = (f, g)$$

for all $f, g \in$ P[0,1]

Again, $(af + bg, h) = \int_0^1 [af(t) + bg(t)] h(t) dt$

$= a \int_0^1 f(t) h(t) dt + b \int_0^1 g(t) h(t) dt$

$= a(f, h) + b(g, h) \;\; for\ all\ a, b \in R\ and\ f, g, h \in$ P[0,1].

So the linear space P[0,1] *is an inner product space*

The norm defined by the inner product is given by

$\|f\| = (f, f)^{\frac{1}{2}} = [\int_0^1 |f(t)|^2 dt]^{\frac{1}{2}}$ *and the metric d is defined by the norm is given by*

$$d(f, g) = \|f - g\| = [\int_0^1 |f(t) - g(t)|^2 dt]^{\frac{1}{2}}$$

But the inner product is not complete

For this ,

$Let\, P_n(x) = \sum_{i=0}^{n} \frac{1}{2^i} x^i$ *then for* $g(x) = \frac{1}{1-\frac{1}{2}x}$ $, 0 \le x \le 1$

$So\, P_n \to g$ *but* $g \notin$ P[0,1]. So (P_n) is a *Cauchy sequence in* P[0,1] *which does not converge to a vector* P[0,1].

Thus P[0,1] *is an inner product space which is not a Hilbert Space.*

Problem: If x *and* y *are any two vectors in a Hilbert Space ,then show that*

$\|x+y\|^2 - \|x-y\|^2 = 4Re(x,y)$

Solution: we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (y,x) \text{...(1)}$$

and $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - (x,y) + (y,x)$ ... (2)

From (1) and (2) we get

$$\|x+y\|^2 - \|x-y\|^2 = (x,y) + (y,x) + (x,y) + (y,x)$$

$$= 2[(x,y) + (y,x)]$$

$$= 2[(x,y) + \overline{(x,y)}]$$

$$= 2[2\mathrm{Re}(x,y)] = 4\mathrm{Re}(x,y).$$

END.