# Boundary Condition, Streamline and Path line (5)

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***M.Sc. Mathematics*** ***Semester:*** *III*

***Paper: Fluid Dynamics*** *XII* *(**MAT CC**-12)*

***Patna University ,Patna***

August 26, 2020

## 1 Boundary Condition

When fluid is in contact with a rigid solid surface(or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact :

**Definition 1.** *The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.*

Let $n$ denote a normal unit vector drawn at the point $P$ of the surface of contact and let $\boldsymbol{q}$ denote the fluid velocity at $P$. When the rigid surface of contact is rest ($\boldsymbol{u} = 0$) in Fig.1a. Then normal velocity $n \cdot \boldsymbol{q} = 0$ at each point of the surface. This express that normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point.

Next, let the rigid surface be in motion and its velocity be $\boldsymbol{u}$ at point $P$ Fig.1b. Then we must have

$$\boldsymbol{u} \cdot n = \boldsymbol{q} \cdot n \quad \text{or} \quad (\boldsymbol{q} - \boldsymbol{u}) \cdot n = 0$$

### 1.1 Conditions at the boundary surface

We propose to derive the differential equation satisfied by a boundary surface of a fluid. Thus, we discuss the problem:

To find the condition that the surface $F(r,t) = 0$ or $F(x,y,z,t) = 0$ may be boundary surface for Fig.1b.

$$\text{Let P be a point on the moving boundary surface} \quad F(r,t) = 0 \tag{1}$$

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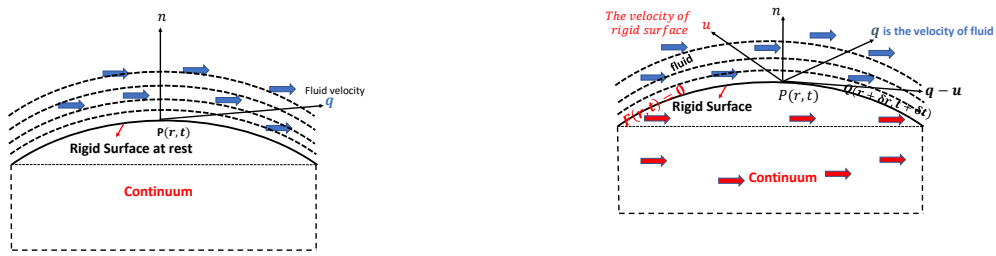


(a) (b)

Figure 1: Boundary condition representation.

where the fluid velocity is $\boldsymbol{q}$ and the velocity of the surface is $\boldsymbol{u}$. Now in order to preserve contact, is given as

$$\boldsymbol{u} \cdot n = \boldsymbol{q} \cdot n \quad \text{or} \quad (\boldsymbol{q} - \boldsymbol{u}) \cdot n = 0 \tag{2}$$

where $n$ in the unit normal vector drawn at point $P$ on the boundary surface equation (1). For direction ratio $n$

$$dF = \left(\frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k\right) \cdot (dxi + dyj + dzk) = 0 \qquad dF = \nabla F \cdot dr = 0$$

$$\nabla F = \left(\frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k\right) \tag{3}$$

Which shows that $\nabla F$ is normal to the boundary surface $F(r, t) = 0$. Hence $n$ is parallel to $\nabla F$ and hence we may write $n = k\nabla F$. Then equation (3) becomes

$$(\boldsymbol{q} - \boldsymbol{u}) \cdot k\nabla F = 0 \quad \Longrightarrow \quad \boldsymbol{u} \cdot \nabla F = \boldsymbol{q} \cdot \nabla F \tag{4}$$

Let $P(r, t)$ moves to a point $Q(r + \delta r, t + \delta t)$, Then point $Q$ must be lies on the boundary surface, it means satisfied the boundary surface (1), at time $t + \delta t$, given by

$$F(r + \delta r, t + \delta t) = 0 \tag{5}$$

RHS of equation (5) expanding by Taylor theorem and neglecting highier order of $\delta r$ and $\delta t$, then equation (5) becomes

$$F(r + \delta r, t + \delta t) = F(r, t) + \delta r \cdot \nabla F + \delta t \frac{\partial F}{\partial t} = 0$$

Using equation (1) then

$$\frac{\partial F}{\partial t} + \frac{\delta r}{\delta t} \cdot \nabla F = 0 \tag{6}$$

$$\lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \boldsymbol{u}$$

Then equation (6) becomes

$$\frac{\partial F}{\partial t} + \boldsymbol{u} \cdot \nabla F = 0 \tag{7}$$

Using equation (4)

$$\frac{\partial F}{\partial t} + \boldsymbol{q} \cdot \nabla F = 0 \tag{8}$$

which is the required condition for $F(r,t) = 0$ to be boundary condition.

**Remark 1.** *Let $\boldsymbol{q} = ui + vj + wk$ then equation (8) may be written as*

$$\frac{\partial F}{\partial t} + (ui + vj + wk) \cdot \left( \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k \right) = 0$$

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \quad or \quad \frac{dF}{dt} = 0 \tag{9}$$

*Equation (9) is the required condition in Cartesian coordinate for $F(x,y,z,t) = 0$ to be boundary surface.*

**Remark 2.** *The normal velocity of the boundary*

$$= \boldsymbol{u} \cdot n = \boldsymbol{u} \cdot \frac{\nabla F}{|\nabla F|} = \frac{-\frac{\partial F}{\partial t}}{\left| \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k \right|}$$

$$= \frac{-\frac{\partial F}{\partial t}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

$$= -\frac{u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

**Remark 3.** *When the boundary surface is at rest, then $\frac{\partial F}{\partial t} = 0$ and hence the condition (9) reduce to*

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \tag{10}$$

Example 1. Show that $(x^2/a^2)\tan^2 t + (y^2/b^2)\cot^2 t = 1$ is possible form for the boundary surface of a liquid, and also find an expression for the normal velocity.

Example 2. Show that $(x^2/a^2k^2t^{2n}) + kt^n(y^2/b^2 + z^2/b^2) = 1$ is possible form for the boundary surface of a liquid, and also find velocity component.

## 1.2 Streamline or line of flow

**Definition 2.** *A streamline is a curve drawn in the fluid so that its tangent at each point is in the direction of motion (i.e., fluid velocity) at that point.*

Let $\boldsymbol{r} = xi + yj + zk$ be the position vector of a point $P$ on a straight line and let $\boldsymbol{q} = ui + vj + wk$ be the fluid velocity at point $P$. Then $\boldsymbol{q}$ is parallel to $\boldsymbol{dr}$ at point $P$ on the streamline. Thus the equation of streamline is given by

$$\boldsymbol{q} \times \boldsymbol{dr} = 0 \tag{11}$$

$$(ui + vj + wk) \times (dxi + dyj + dzk) = 0$$

$$(vdz - wdy)i + (wdx - udz)j + (udy - vdx)k = 0i + 0j + 0k = 0$$

hence

$$vdz = wdy\ , \qquad wdx = udz, \qquad udy = vdx$$

$$\frac{dx}{u} = \frac{dy}{y} = \frac{dz}{w} \tag{12}$$

Equation (12) represent system of linear differential equation. So solution of equation (12) has double infinite set of solutions.

## 1.3 Path line or Path of the particle

**Definition 3.** *A path line is the curve or trajectory along which a particular fluid particle travel during its motion.*

$$\text{The differential equation of a path line is} \qquad \frac{\boldsymbol{dr}}{dt} = \boldsymbol{q} \tag{13}$$

so that

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt} \tag{14}$$

where $\boldsymbol{q} = ui + vj + wk$ and $\boldsymbol{r} = xi + yj + zk$

**Remark 4.** *Let a fluid particle of fixed identity be at $(x_0, y_0, z_0)$ when $t = t_0$ then path line determine from equations*

$$\left.\begin{aligned} \tfrac{dx}{dt} &= u(x, y, z, t) \\ \tfrac{dy}{dt} &= v(x, y, z, t) \\ \tfrac{dz}{dt} &= w(x, y, z, t) \end{aligned}\right\} \tag{15}$$

*with initial conditions*

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0$$

All the best...

Next in $6^{\text{th}}$ Econtent