DUAL PROBLEM
(M.Sc. Sem-III)

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1. PARAMETRIC LINEAR PROGRAMMING

Parametric linear programming is an extension of the post-optimal analysis presented. It investigates the effect of predetermined continuous variations in the objective function coefficients and the right-hand side of the constraints on the optimum solution.

Let $X = (x_1, x_2, \ldots.., x_n)$ and define the LP as

$$\text{Maximize } z = \left\{ CX \mid \sum_{j=1}^{n} P_j x_j = b,\ X \geq 0 \right\}$$

In parametric analysis, the objective function and right-hand side vectors, C and b, are replaced with the parameterized functions C(t) and b(t), where t is the parameter of variation. Mathematically, t can assume any positive or negative value. In practice, however, t usually represents time, and hence it is non-negative. In this presentation we will assume $t \geq 0$.

Parametric Changes in C

Let $X_{B_i}, B_i, C_{B_i}(t)$ be the elements that define the optimal solution associated with critical t_i (the computations start at $t_0 = 0$ with B_0 as its optimal basis). Next, the critical value t_{i+1} and its optimal basis, if one exists, is determined. Because changes in C can affect only the optimality of the problem, the current solution $X_{B_i} = B_i^{-1}b$ will remain optimal for some $t \geq t_i$ so long as the reduced cost, $z_j(t) - c_j(t)$, satisfies the following optimality condition :

$$z_j(t) - c_j(t) = C_{B_i}(t) B_i^{-1} P_j - c_j(t) \geq 0, \text{ for all } j$$

The value of t_{i+1} equals the largest $t > t_i$ that satisfies all the optimality conditions.

Note that nothing in inequalities requires $C(t)$ to be linear in t. Any function $C(t)$, linear or non-linear, is acceptable. However, with non-linearity the numerical manipulation of the resulting inequalities may be cumbersome.

Example-1

$$\text{Maximize } z = (3-6t)x_1 + (2-2t)x_2 + (5+5t)x_3$$

subject to

$$x_1 + 2x_2 + x_3 \leq 40$$
$$3x_1 \quad\ + 2x_3 \leq 60$$
$$x_1 + 4x_2 \qquad \leq 30$$
$$x_1, x_2, x_3 \geq 0$$

We have

$$C(t) = (3-6t,\ 2-2t,\ 5+5t),\ t \geq 0$$

The variables x_4, x_5 and x_6 will be used as the slack variables associated with the three constraints.

Optimal Solution at $t = t_0 = 0$

$Basic$	x_1	x_2	x_3	x_4	x_5	x_6	$Solution$
z	4	0	0	1	2	0	160
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	5
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	30
x_6	2	0	0	-2	1	1	10

$$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$C_{B_0}(t) = (2-2t, 5+5t, 0)$$

$$B_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

The optimality conditions for the current non-basic vectors, P_1, P_4 and P_5, are

$$\left\{C_{B_0}(t)B_0^{-1}P_j - c_j(t)\right\}_{j=1,4,5} = (4+14t, 1-t, 2+3t) \geq 0$$

Thus, X_{B_0} remains optimal so long as the following conditions are satisfied :

$$4+14t \geq 0$$
$$1-t \geq 0$$
$$2+3t \geq 0$$

Because $t \geq 0$, the second inequality gives $t \leq 1$ and the remaining two inequalities are satisfied for all $t \geq 0$. We thus have $t_1 = 1$, which means that X_{B_0} remains optimal (and feasible) for $0 \leq t \leq 1$.

The reduced cost $z_4(t) - c_4(t) = 1-t$ equals zero at $t = 1$ and becomes negative for $t > 1$. Thus, P_4 must enter the basis for $t > 1$. In this case, P_2 must leave the basis. The new basic solution X_{B_1} is the alternative solution obtained at $t = 1$ by letting P_4 enter the basis – that is, $X_{B_1} = (x_4, x_3, x_6)^T$ and $B_1 = (P_4, P_3, P_6)$.

Alternative Optimal Basis at $t = t_1 = 1$

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_1^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$X_{B_1} = (x_4, x_3, x_6)^T = B_1^{-1}b = (10, 30, 30)^T$$

$$C_{B_1}(t) = (0, 5+5t, 0)$$

The associated non-basic vectors are P_1, P_2, and P_5, and we have

$$\left\{C_{B_1}(t)B_1^{-1}P_j - c_j(t)\right\}_{j=1,\,2,\,5} = \left(\frac{9+27t}{2},\ -2+2t,\ \frac{5+5t}{2}\right) \geq 0$$

According to these conditions, the basic solution x_{B_1} remains optimal for all $t \geq 1$ Observe that the optimality condition, $-2+2t \geq 0$, automatically "remembers" that X_{B_1} is optimal for a range of *t* that starts from the last critical value $t_1 = 1$. This will always be the case in parametric programming computations.

The optimal solution for the entire range of *t* is summarized below. The value of *z* is computed by direct substitution.

t	x_1	x_2	x_3	z
$0 \leq t \leq 1$	0	5	30	$160+140t$
$t \geq 1$	0	0	30	$150+150t$

Parametric Changes in b

The parameterized right-hand side $b(t)$ can affect only the feasibility of the problem.

The critical values of *t* are thus determined from the following condition :

$$X_B(t) = B^{-1}b(t) \geq 0$$

Example-2

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 40 - t \\ 3x_1 \quad + 2x_3 &\leq 60 + 2t \\ x_1 + 4x_2 \quad &\leq 30 - 7t \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

Assume that $t \geq 0$.

At $t = t_0 = 0$. We thus have

$$X_{B_0} = (x_2,\ x_3,\ x_6)^T = (5,\ 30,\ 10)^T$$

$$B_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

To determine the first critical value t_1, we apply the feasibility conditions $X_{B_0}(t) = B_0^{-1}b(t) \geq 0$, which yields

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 5-t \\ 30+t \\ 10-3t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These inequalities are satisfied for $t \le \frac{10}{3}$, meaning that $t_1 = \frac{10}{3}$ and that the basis B_0 remains feasible for the range $0 \le t \le \frac{10}{3}$. However, the values of the basic variables x_2, x_3, and x_6 will change with t as given above.

The value of the basic variable $x_6\left(=10-3t\right)$ will equal zero at $t = t_1 = \frac{10}{3}$, and will become negative for $t > \frac{10}{3}$. Thus, at $t = \frac{10}{3}$, we can determine the alternative basis B_1 by applying the revised dual simplex method. The leaving variable is x_6.

Alternative Basis at $t = t_1 = \frac{10}{3}$

Given that x_6 is the leaving variable, we determine the entering variable as follows :

$$X_{B_0} = \left(x_2,\, x_3,\, x_6\right)^T,\; C_{B_0} = \left(2,\, 5,\, 0\right)$$

Thus,

$$\left\{z_j - c_j\right\}_{j=1,\,4,\,5} = \left\{C_{B_0} B_0^{-1} P_j - c_j\right\}_{j=1,\,4,\,5} = \left(4,\, 1,\, 2\right)$$

Next, for non-basic x_j, $j = 1,\, 4,\, 5,$ we compute

$$\begin{aligned}\left(\text{Row of } B_0^{-1} \text{ associated with } x_6\right)\left(P_1,\, P_4,\, P_5\right) &= \left(\text{Third row of } B_0^{-1}\right)\left(P_1,\, P_4,\, P_5\right)\\ &= \left(-2,\, 1,\, 1\right)\left(P_1,\, P_4,\, P_5\right)\\ &= \left(2,\, -2,\, 1\right)\end{aligned}$$

The entering variable is thus associated with

$$\theta = \min\left\{-,\, \left|\frac{1}{-2}\right|,\, -\right\} = \frac{1}{2}$$

Thus, P_4 is the entering vector. The alternative basic solution and its B_1 and B_1^{-1} are

$$X_{B_1} = \left(x_2,\, x_3,\, x_4\right)^T$$

$$B_1 = \left(P_2,\, P_3,\, P_4\right) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix},\; B_1^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

The next critical value t_2 is determined from the feasibility conditions, $X_{B_1}\left(t\right) = B_1^{-1} b\left(t\right) \ge 0,$

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{30-7t}{4} \\ 30+t \\ \frac{-10+3t}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These conditions show that B_1 remains feasible for $\frac{10}{3} \le t \le \frac{30}{7}$.

At $t = t_2 = \frac{30}{7}$, an alternative basis can be obtained by the revised dual simplex method. The leaving variable is x_2, because it corresponds to the condition yielding the critical value t_2.

Alternative Basis at $t = t_2 = \frac{30}{7}$

Given that x_2 is the leaving variable, we determine the entering variable as follows :

$$X_{B_1} = (x_2, x_3, x_4)^T, \; C_{B_1} = (2, 5, 0)$$

Thus,

$$\{z_j - c_j\}_{j=1,5,6} = \{C_{B_1} B_1^{-1} P_j - c_j\}_{j=1,5,6} = \left(5, \frac{5}{2}, \frac{1}{2}\right)$$

Next, for non-basic x_j, $j = 1, 5, 6$, we compute

$$\begin{aligned}\left(\text{Row of } B_1^{-1} \text{ associated with } x_2\right)(P_1, P_5, P_6) &= \left(\text{First row of } B_1^{-1}\right)(P_1, P_5, P_6)\\ &= \left(0, 0, \frac{1}{4}\right)(P_1, P_5, P_6)\\ &= \left(\frac{1}{4}, 0, \frac{1}{4}\right)\end{aligned}$$

Because all the denominator elements, $\left(\frac{1}{4}, 0, \frac{1}{4}\right)$, are ≥ 0, the problem has no feasible solution for $t > \frac{30}{7}$ and the parametric analysis ends at $t = t_2 = \frac{30}{7}$.

The optimal solution is summarized as

t	x_1	x_2	x_3	z
$0 \le t \le \frac{10}{3}$	0	$5 - t$	$30 + t$	$160 + 3t$
$\frac{10}{3} \le t \le \frac{30}{7}$	0	$\frac{30 - 7t}{4}$	$30 + t$	$165 + \frac{3}{2}t$
$t > \frac{30}{7}$	(No feasible solution exists)			

PROBLEM SET

1. In Example-2, find the first critical value, t_1, and define the vectors of B_1 in each of the following cases :

   (a) $b(t) = (40 + 2t, 60 - 3t, 30 + 6t)^T$

   (b) $b(t) = (40 - t, 60 + 2t, 30 - 5t)^T$

2. Study the variation in the optimal solution of the following parameterized LP, given $t \geq 0$.

$$\text{Minimize } z = 4x_1 + x_2 + 2x_3$$

subject to

$$3x_1 + x_2 + 2x_3 = 3 + 3t$$

$$4x_1 + 3x_2 + 2x_3 \geq 6 + 2t$$

$$x_1 + 2x_2 + 5x_3 \leq 4 - t$$

$$x_1,\ x_2,\ x_3 \geq 0$$

3. The analysis in this section assume that the optimal LP solution at $t = 0$ is obtained by the (primal) simplex method. In some problems, it may be more convenient to obtain the optimal solution by the dual simplex method. Show how the parametric analysis can be carried out in this case, and then analyze the LP of example, assuming that $t \geq 0$ and the right-hand side vector is :

$b(t) = (3 + 2t,\ 6 - t,\ 3 - 4t)^T$

4. Solve problem-2 assuming that the right-hand side is changed to

$$b(t) = \left(3 + 3t^2,\ 6 + 2t^2,\ 4 - t^2\right)^T$$

Further assume that t can be positive, zero, or negative