e-content

BY

DR. ABHAY KUMAR (GUEST FACULTY)

P.G. Department of mathematics

Patna university, patna

sem-3 (Functional analysis)

Topic: Theorem based on inner product spaces and Hilbert spaces.

Theorem: *Let E be an inner product space over a field K.*

If a norm on E is defined by

$$\|x\| = +\sqrt{(x,x)} = +(x,x)^{\frac{1}{2}} \quad \forall \ x \in E$$

Then E is a normed linear space.

Thus every inner product space is a normed linear space.

Proof: Since $(x,x) \geq 0 \Rightarrow \sqrt{(x,x)} \geq 0 \Rightarrow \|x\| \geq 0$

Also

$$\|x\| = 0 \Leftrightarrow (x,x) = 0 \Leftrightarrow x = 0$$

Again $\|\alpha x\|^2 = (\alpha x, \alpha x) = \alpha(x, \alpha x) = \alpha\bar{\alpha}(x, x) =$ $|| \alpha\|^2 \|x\|^2$

$\|\alpha x\| = \|\alpha\| \|x\|$

We have to prove the last condition of the normed

Linear space for this we first prove that

$|\mathrm{Re}(x, y)| \leq \|x\| \|y\|.\ \forall\ x, y \in E$ *where* $|\mathrm{Re}(x, y)|$ *denotes the real part of* (x, y).

Since for all $x, y \in E$ *and* $\beta \in K$ *we have*

$$0 \leq (x + \beta y, x + \beta y) = (x, x) + \beta(y, x) + \bar{\beta}(x, y)$$
$$+\beta\bar{\beta}(y, y)$$
$$= \|x\|^2 + \beta\overline{(x, y)} + \bar{\beta}(x, y) + ||\ \beta\|^2 \|y\|^2$$

For real β *we have*

$$\|x\|^2 + \beta[\overline{(x, y)} + (x, y)] + ||\ \beta\|^2 \|y\|^2 \geq 0$$

$$\|x\|^2 + 2\mathrm{Re}(\mathrm{x}, \mathrm{y})\beta + ||\ \beta\|^2 \|y\|^2 \geq 0$$

Putting $a = \|y\|^2$, $b = 2\mathrm{Re}(x, y)$, $c = \|x\|^2$ *we have*

$$a\beta^2 + b\beta + c \geq 0$$

This implies that $b^2 \leq 4ac$

$So\ [2Re(x,y)]^2 \leq 4\|x\|^2 . \|y\|^2$

$Hence\ |\mathrm{Re}(x,y)| \leq \|x\|\|y\|. \quad ....(1)$

$Now\ we\ have\ \|x+y\|^2 = (x+y, x+y)$

$$= (x,x) + (x,y) + (y,x) + (y,y)$$

$$= (x,x) + (x,y) + \overline{(x,y)} + (y,y)$$

$$= (x,x) + 2Re(x,y) + (y,y)$$

$$= \|x\|^2 + 2\mathrm{Re}(\mathrm{x},\mathrm{y}) + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad [\text{ from } (1)]$$

$$= [\|x\| + \|y\|]^2$$

$\| x+y\| \leq \|x\| + \|y\|.$

Hence all the conditions of the the normed linear space is satisfied hence E is a normed linear space

with respect to the norm $\|x\| = +\sqrt{(x,x)} = +(x,x)^{\frac{1}{2}}$ $\forall\ x \in E$ *defined on E.*

Def(Hilbert space):Let E be an inner product space

And let a norm on E be defined by $\|x\| = +\sqrt{(x,x)} = +(x,x)^{\frac{1}{2}} \quad \forall\ x \in E$ *.Let d be the metric on E defined by* $d(x,y)=\|x-y\|\ \forall\ x \in E$ *If* (E,d) *is a complete metric space then E is said to be a Hilbert space.*

Thus every Hilbert space is a Banach space

Example: The real linear space R^n *is a Hilbert space With respect to the inner product defined by*

$$(x,y) = \sum_{i=1}^{n} x_i y_i$$

$for\ x = (x_1, x_2 \dots x_n), y = (y_1, y_2, \dots y_n) \in R^n$

The norm defined by the inner product is given by

$\|x\| = (x,x)^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ *and metric d is defined by*

$d(x,y) = \|x-y\| = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$ then (R^n, d)

Is a complete metric space and hence R^n *is a real Hilbert space.*

END.