Equation of continuity by Lagrangian Approach (4)

Binod Kumar*
M.Sc. Mathematics ***Semester:*** *III*
Paper: Fluid Dynamics *XII* *(**MAT CC**-12)*
Patna University ,Patna

August 21, 2020

1 The equation of continuity by Lagrangian Method

Let R_0 be the region occupied by portion of a fluid at time $t = 0$, and R the region occupied by the same fluid at any time t.

Let (a, b, c) be the initial co-ordinate of a fluid particle P_0 enclosed in this element and ρ_0 is the density.

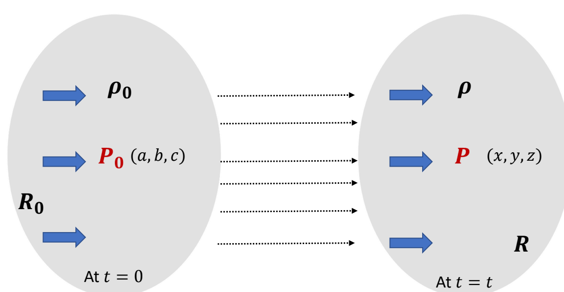

Figure 1: Fluid flow in Lagrangian form.

Then mass of the fluid element at $t = 0$ is $\rho_0 \delta a \delta b \delta c$

Let P be the subsequent position of P_0 at time t and let ρ be density of the fluid here.

*Corresponding author, e-mail:binodkumararyan@gmail.com, Telephone: +91-9304524851

Then mass of the fluid element at $t = t$ is $\rho_0 \delta x \delta y \delta z$

From law of conservation of mass

$$\iiint_{R_0} \rho_0 \delta a \delta b \delta c - \iiint_{R} \rho_0 \delta x \delta y \delta z = 0 \tag{1}$$

We know that

$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$\delta x \delta y \delta z = \boldsymbol{J} \delta a \delta b \delta c \tag{2}$$

$$\rho_0 - \boldsymbol{J} \rho = 0 \tag{3}$$

Which is required equation of continuity in Lagrangian approach.

Theorem 1. *Equivalence between Euler and Lagrangian form of equations of continuity.*

Proof. Let R_0 be the region occupied by portion of a fluid at time $t = 0$, and R the region occupied by the same fluid at any time t. Let (a, b, c) be the initial co-ordinate of a fluid particle P_0 enclosed in this element and ρ_0 is the density. Then mass of the fluid element at $t = 0$ is $\rho_0 \delta a \delta b \delta c$ Let P be the subsequent position of P_0 at time t and let ρ be density of the fluid here. Then mass of the fluid element at $t = t$ is $\rho_0 \delta x \delta y \delta z$

The velocity components in the two systems are given by

$$u = dx/dt, \qquad v = dy/dt, \qquad w = dz/dt$$

and also,

$$x = x(a,b,c,t), \qquad y = y(a,b,c,t), \qquad z = z(a,b,c,t)$$

$$\therefore \quad \frac{\partial u}{\partial a} = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) \quad \text{So that} \quad \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial u}{\partial a}$$

Similarly above result for the velocity $\boldsymbol{q} = (u, v, w)$

For velocity component u

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial u}{\partial a} \\ \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) = \frac{\partial u}{\partial b} \\ \frac{d}{dt}\left(\frac{\partial z}{\partial a}\right) = \frac{\partial u}{\partial c} \end{array}\right\} \tag{4}$$

For velocity component v

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial v}{\partial a} \\ \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) = \frac{\partial v}{\partial b} \\ \frac{d}{dt}\left(\frac{\partial z}{\partial a}\right) = \frac{\partial v}{\partial c} \end{array}\right\} \tag{5}$$

For velocity component w

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial w}{\partial a} \\ \frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) = \frac{\partial w}{\partial b} \\ \frac{d}{dt}\left(\frac{\partial z}{\partial a}\right) = \frac{\partial w}{\partial c} \end{array}\right\} \tag{6}$$

The equation of continuity in the Lagrangian form is

$$\rho \boldsymbol{J} = \rho_0 \tag{7}$$

where

$$J = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} \tag{8}$$

Also. the equation of continuity in the Eulerian form is

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{9}$$

Differentiating equation (7) w.r.t. t and using equations (4), (5) and (6), we get

$$\frac{d\boldsymbol{J}}{dt} = \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \end{vmatrix} \tag{10}$$

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3 \tag{11}$$

where $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and $\boldsymbol{J}_3$ are represent first, second and third term of R.H.S. of equation (10)

For $\boldsymbol{J}_1$, Since

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}$$

and

$$\frac{\partial u}{\partial c} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c}$$

Then $\boldsymbol{J}_1$ can be written as (using concept Deteminant$(\boldsymbol{J}_1^{\mathrm{T}})$ = Deteminant$\boldsymbol{J}_1$)). Thus, we have

$$\boldsymbol{J}_1 = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

$$\boldsymbol{J}_1 = \frac{\partial u}{\partial x}\boldsymbol{J} \tag{12}$$

Similarly,

$$\boldsymbol{J}_2 = \frac{\partial u}{\partial x}\boldsymbol{J} \quad \text{and} \quad \boldsymbol{J}_3 = \frac{\partial u}{\partial x}\boldsymbol{J}$$

$\therefore$ equation (11) becomes

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{J}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{13}$$

- **Derivation of Eulerian form from Lagrangian form**

  Lagrangian equation of continuity (7)

$$\frac{d(\rho\boldsymbol{J})}{dt} = \frac{d\rho_0}{dt} = 0$$

$$\boldsymbol{J}\frac{d\rho}{dt} + \rho\frac{d\boldsymbol{J}}{dt} = 0$$

$$\implies \quad \boldsymbol{J}\frac{d\rho}{dt} + \rho\boldsymbol{J}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad \text{using (13)}$$

$$\therefore \quad \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{d\rho}{dt} + \rho\left(\nabla\cdot\boldsymbol{q}\right) = 0 \tag{14}$$

  Which is required Eulerian equation of continuity (14).

- **Derivation of Lagrangian form from Eulerian form**

  Equation (14) can be written with the help of equatipn (13)

$$\frac{d\rho}{dt} + \frac{\rho}{\boldsymbol{J}}\frac{d\boldsymbol{J}}{dt} = 0$$

$$\frac{d}{dt}\left(\rho\boldsymbol{J}\right) = 0$$

$$\implies \quad \int \frac{d}{dt}\left(\rho\boldsymbol{J}\right)dt = \int (0)dt \quad \therefore \quad \rho\boldsymbol{J} = \rho_0 \;\; \text{(Constant)} \;\; \text{say}$$

□

All the best...

Next in 5$^{\text{th}}$ Econtent