# DUAL PROBLEM <br> (M.Sc. Sem-III) 

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## 1. DEFINITION OF THE DUAL PROBLEM

The dual problem is an LP defined directly and systematically from the primal (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints $\mathrm{d}(\leq, \geq$, or $=$ ), and orientation of the variables (non-negative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a single definition that automatically subsumes all forms of the primal.

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

Rules for Constructing the Dual Problem

| Primal problem <br> objective $^{\mathrm{n}}$ | Objective | Constraints type | Variables sign |
| :---: | :---: | :---: | :---: |
|  | Coblem |  |  |
|  | Maximization | $\geq$ | Unrestricted |
| Minimization | Minimization | $\leq$ | Unrestricted |

* All primal constraints are equations with non-negative right-hand side and all the variables armon-negative.

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint $(\leq, \geq$, or $=)$, and the sign of the dual variables are summarized in above table. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$ ) is that if the dual objective is minimization (i.e., pointing down), then the constraints are all of the type $\geq$ (i.e., pointing up). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in above and also show that our definition incorporates all forms of the primal automatically.

## Example-1.1

## Primal

$$
\begin{gathered}
\text { Maximize } z=5 x_{1}+12 x_{2}+4 x_{3} \\
\text { subject to } \\
x_{1}+2 x_{2}+x_{3} \leq 10 \\
2 x-x_{2}+3 x_{3}=8 \\
x_{1}, x_{2}, x_{3} \geq 0
\end{gathered}
$$

## Primal in equation form

$$
\text { Maximize } z=5 x_{1}+12 x_{2}+4 x_{3}+0 x_{4}
$$

subject to

$$
\begin{array}{ll}
x_{1}+2 x_{2}+x_{3}+x_{4}=10 & y_{1} \\
2 x_{1}-x_{2}+3 x_{3}+0 x_{4}=8 & y_{2}
\end{array}
$$

$$
x_{1}, x_{2}, x_{3}, x_{4} \geq 0
$$

## Dual problem

$$
\text { Minimize } w=10 y_{1}+8 y_{2}
$$

subject to

$$
\left.\begin{array}{l}
y_{1}+2 y_{2} \geq 5 \\
2 y_{1}-y_{2} \geq 12 \\
y_{1}+3 y_{2} \geq 4 \\
\quad y_{1}+0 y_{2} \geq 0 \\
y_{1}, y_{2} \text { unrestricted }
\end{array}\right\} \Rightarrow\left(y_{1} \geq 0, y_{2} \text { unrestricted }\right)
$$

## Example-1.2

## Primal

Minimize $z=15 x_{1}+12 x_{2}$
subject to

$$
\begin{gathered}
x_{1}+2 x_{2} \geq 3 \\
2 x_{1}-4 x_{2} \leq 5 \\
x_{1}, x_{2} \geq 0
\end{gathered}
$$

## Primal in equation form

Minimize $z=15 x_{1}+12 x_{2}+0 x_{3}+0 x_{4}$
subject to

$$
\begin{gathered}
x_{1}+2 x_{2}-x_{3}+0 x_{4}=3 \\
2 x_{1}-4 x_{2}+0 x_{3}+x_{4}=5 \\
x_{1}, x_{2}, x_{3}, x_{4} \geq 0
\end{gathered}
$$

Dual variables
subject to

$$
\left.\begin{array}{l}
y_{1}+2 y_{2} \leq 15 \\
2 y_{1}-4 y_{2} \leq 12 \\
-y_{1} \quad \leq 0 \\
y_{2} \leq 0 \\
y_{1}, y_{2} \text { unrestricted }
\end{array}\right\} \Rightarrow\left(y_{1} \geq 0, y_{2} \leq 0\right)
$$

## Example 1.3

## Primal

## Primal in equation form

Dual variables
Substitute $x_{1}=x_{1}^{+}-x_{1}^{-}$
Maximize $z=5 x_{1}+6 x_{2}$

$$
\begin{gathered}
\text { subject to } \\
x_{1}+2 x_{2}=5 \\
-x_{1}+5 x_{2} \geq 3 \\
4 x_{1}+7 x_{2} \leq 8
\end{gathered}
$$

$x_{1}$ unrestricted, $x_{2} \geq 0$

$$
\text { Maximize } z=5 x_{1}^{+}-5 x_{1}^{-}+6 x_{2}
$$

## Dual Problem

$$
\begin{array}{cc}
x_{1}^{-}-x_{1}^{+}+2 x_{2}=5 & y_{1} \\
-x_{1}^{-}+x_{1}^{+}+5 x_{2}-x_{3}=3 & y_{2} \\
4 x_{1}^{-}-4 x_{1}^{+}+7 x_{2}+x_{4}=8 & y_{3} \\
x_{1}^{-}, x_{1}^{+}, x_{2}, x_{3}, x_{4} \geq 0 &
\end{array}
$$

$$
\text { Minimize } z=5 y_{1}+3 y_{2}+8 y_{3}
$$

subject to

$$
\begin{array}{r}
\left.\begin{array}{r}
y_{1}-y_{2}+4 y_{3} \geq 5 \\
-y_{1}+y_{2}-4 y_{3} \geq-5
\end{array}\right\} \Rightarrow\left(y_{1}-y_{2}+4 y_{3}=5\right) \\
2 y_{1}+5 y_{2}+7 y_{3} \geq 6
\end{array}
$$

$$
\left.\begin{array}{r}
-y_{2} \geq 0 \\
y_{3} \geq 0 \\
y_{1}, y_{2}, y_{3} \text { unrestricted }
\end{array}\right\} \Rightarrow\left(y_{1} \text { unrestricted, } y_{2} \leq 0, y_{3} \geq 0\right)
$$

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.
Summary of the Rules for Constructing the Dual. The general conclusion from the preceding example is that the variables and constraints in the primal and dual problems are defined by the rules in Table 1.3. It is a good exercise to verity that these explicit rules are subsumed by the general rules in Table 1.2.

Table 1.4 Rules for Constructing the Dual Problem

| Maximization problem |  | Minimization problem |
| :---: | :---: | :---: |
| Constraints | $\Leftrightarrow$ | $\leq 0$ |
| $\geq$ | $\Leftrightarrow$ | $\geq 0$ |
| $=$ | $\Leftrightarrow$ | Unrestricted |
| Variables |  | Constraints |
| $\geq 0$ | $\Leftrightarrow$ | $\geq$ |
| $\leq 0$ | $\Leftrightarrow$ | $\leq$ |
| Unrestricted | $\Leftrightarrow$ | $=$ |
|  |  |  |

1. Write the dual for each of the following primal problems :
(a) Maximize $z=-5 x_{1}+2 x_{2}$
subject to

$$
\begin{aligned}
& -x_{1}+x_{2} \leq-2 \\
& 2 x_{1}+3 x_{2} \leq 5 \\
& x_{1}, x_{2} \geq 0
\end{aligned}
$$

(b) Minimize $z=6 x_{1}+3 x_{2}$
subject to

$$
\begin{aligned}
& 6 x_{1}-3 x_{2}+x_{3} \geq 2 \\
& 3 x_{1}+4 x_{2}+x_{3} \geq 5 \\
& x_{1}, x_{2}, x_{3} \geq 0
\end{aligned}
$$

(c) Maximize $z=x_{1}+x_{2}$
subject to

$$
\begin{aligned}
& 2 x_{1}+x_{2}=5 \\
& 3 x_{1}-x_{2}=6 \\
& x_{1}, x_{2} \text { unrestricted }
\end{aligned}
$$

## 2. Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints.

## Method 1.

$\binom{$ Optimal value of }{ dual variable $y_{i}}=\left(\begin{array}{c}\text { Optimal primal z-coefficient of starting variable } x_{i} \\ + \\ \text { Original objective coefficient of } x_{i}\end{array}\right)$

## Method 2.

$\binom{$ Optimal value of }{ dual variable }$=\left(\begin{array}{c}\text { Row vector of } \\ \text { original objective coefficients } \\ \text { of optimal primal basic variables }\end{array}\right) \times\binom{$ Optimal primal }{ inverse }
The elements of the row vector must appear in the same order in which the basic variables are listed in the Basic column of the simplex tableau.

## Example 1.4

Consider the following LP :

$$
\text { Maximize } z=5 x_{1}+12 x_{2}+4 x_{3}
$$

subject to

$$
\begin{aligned}
& x_{1}+2 x_{2}+x_{3} \leq 10 \\
& 2 x_{1}-x_{2}+3 x_{3}=8 \\
& x_{1}, x_{2}, x_{3} \geq 0
\end{aligned}
$$

To prepare the problem for solution by the simplex method, we add a slack $x_{4}$ in the first constraint and an $\operatorname{artificial} R$ in the second. The resulting primal and the associated dual problems are thus defined as follows :

## Primal

## Dual

$$
\text { Maximize } z=5 x_{1}+12 x_{2}+4 x_{3}-M R
$$

$$
\begin{gathered}
x_{1}+2 x_{2}+x_{3}+x_{4}=10 \\
2 x_{1}-x_{2}+3 x_{3}+R=8 \\
x_{1}, x_{2}, x_{3}, x_{4}, R \geq 0
\end{gathered}
$$

$$
\text { Minimize } w=10 y_{1}+8 y_{2}
$$

subject to
subject to

$$
\begin{gathered}
y_{1}+2 y_{2} \geq 5 \\
2 y_{1}-y_{2} \geq 12 \\
y_{1} \geq 0
\end{gathered}
$$

$$
y_{2} \geq-M\left(\Rightarrow y_{2} \text { unrestricted }\right)
$$

We not show how the optimal dual values are determined using the two methods described at the start of this section.

Method-1: The starting primal variables $x_{4}$ and $R$ uniquely correspond to the dual variables $y_{1}$ and $y_{2}$, respectively. Thus, we determine the optimum dual solution as follows :

Starting primal basic variables
$z$-equation coefficients

| $x_{4}$ | $R$ |
| :--- | :--- |
| $\frac{29}{5}$ | $-\frac{2}{5}+M$ |
| 0 | $-M$ |
| $y_{1}$ | $y_{2}$ |

Optimal dual values

$$
\frac{29}{5}+0=\frac{29}{5}
$$

$$
-\frac{2}{5}+M+(-M)=-\frac{2}{5}
$$

Method-2 : The optimal inverse matrix, highlighted under the starting variables $x_{4}$ and $R$, is given as

$$
\text { Optimal inverse }=\left(\begin{array}{cc}
\frac{2}{5} & -\frac{1}{5} \\
\frac{1}{5} & \frac{2}{5}
\end{array}\right)
$$

First, we note that the optimal primal variables are listed in the tableu in row order as $x_{2}$ and then $x_{1}$. This means that the elements of the original objective coefficients for the two variables must appear in the same order - namely,
(Original objective coefficients) $=\left(\right.$ Coefficient of $x_{2}$, coefficient of $\left.x_{1}\right)$

$$
=(12,5)
$$

TABLE Optimal Tableau of the Primal of Example

| Basic | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $R$ | Solution |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $z$ | 0 | 0 | $\frac{3}{5}$ | $\frac{29}{5}$ | $-\frac{2}{5}+M$ | $54 \frac{4}{5}$ |
| $x_{2}$ | 0 | 1 | $-\frac{1}{5}$ | $\frac{2}{5}$ | $-\frac{1}{5}$ | $\frac{12}{5}$ |
| $x_{1}$ | 1 | 0 | $\frac{7}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{26}{5}$ |

Thus, the optimal dual values are computed as

$$
\begin{aligned}
\left(y_{1}, y_{2}\right) & =\binom{\text { Original objective }}{\text { coefficients of } x_{2}, x_{1}} \times(\text { Optimal inverse }) \\
& =(12,5)\left(\begin{array}{cc}
\frac{2}{5} & -\frac{1}{5} \\
\frac{1}{5} & \frac{2}{5}
\end{array}\right)=\left(\frac{29}{5},-\frac{2}{5}\right)
\end{aligned}
$$

