

M.S c Mathematics –SEM 3 Rigid Dynamics

CC-13 Unit 1

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Kinetic energy of a rigid body and obtain an expression for it.

Suppose a dynamical system in which the co-ordinates of all particles are expressed in terms of n independent generalised co-ordinates b
the following relation

$$x_i = f(q_1, q_2, \dots, q_n, t)$$

$$y_i = g(q_1, q_2, \dots, q_n, t)$$

$$z_i = h(q_1, q_2, \dots, q_n, t)$$

The velocity components of a particle of mass m are given by

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1} q_1 + \frac{\partial x_i}{\partial q_2} q_2 + \dots + \frac{\partial x_i}{\partial q_n} q_n + \frac{\partial x_i}{\partial t}$$

$$\begin{aligned}
&= \sum_{r=1}^n \sum_{s=1}^n \frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_s} \dot{q}_r \dot{q}_s \\
&\quad + 2 \sum_{r=1}^n \frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_t} \dot{q}_r + \left(\frac{\partial x_i}{\partial t} \right)^2 \\
\dot{y}_i^2 &= \sum_{r=1}^n \sum_{s=1}^n \frac{\partial y_i}{\partial q_r} \frac{\partial y_i}{\partial q_s} \dot{q}_r \dot{q}_s \\
&\quad + 2 \sum_{r=1}^n \frac{\partial y_i}{\partial q_r} \frac{\partial y_i}{\partial q_t} \dot{q}_r + \left(\frac{\partial y_i}{\partial t} \right)^2 \\
\dot{z}_i^2 &= \sum_{r=1}^n \sum_{s=1}^n \frac{\partial z_i}{\partial q_r} \frac{\partial z_i}{\partial q_s} \dot{q}_r \dot{q}_s \\
&\quad + 2 \sum_{r=1}^n \frac{\partial z_i}{\partial q_r} \frac{\partial z_i}{\partial q_t} \dot{q}_r + \left(\frac{\partial z_i}{\partial t} \right)^2
\end{aligned}$$

Putting the values of \dot{x}_i^2 , \dot{y}_i^2 , \dot{z}_i^2 in equation(i) we get

$$\begin{aligned}
2T &= \sum_{i=1}^n m_i \left[\sum_{r=1}^n \sum_{s=1}^n \frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_s} \right. \\
&\quad + \frac{\partial y_i}{\partial q_s} \frac{\partial y_i}{\partial q_s} + \frac{\partial z_i}{\partial q_s} \frac{\partial z_i}{\partial q_s}) \dot{q}_r \dot{q}_s \\
&\quad + 2 \sum_{r=1}^n \left(\frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_t} + \frac{\partial y_i}{\partial q_r} \frac{\partial y_i}{\partial q_t} \right. \\
&\quad \left. \left. + \frac{\partial z_i}{\partial q_r} \frac{\partial z_i}{\partial q_t} \right) \dot{q}_r \right. \\
&\quad \left. + \left(\frac{\partial x_i}{\partial t} \right)^2 + \left(\frac{\partial y_i}{\partial t} \right)^2 + \left(\frac{\partial z_i}{\partial t} \right)^2 \right] \\
&= \sum_{r=1}^n \sum_{s=1}^n \left[\sum_{i=1}^n m_i \left[\frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_s} \right. \right. \\
&\quad \left. \left. + \frac{\partial y_i}{\partial q_s} \frac{\partial y_i}{\partial q_s} + \frac{\partial z_i}{\partial q_s} \frac{\partial z_i}{\partial q_s} \right) \dot{q}_r \dot{q}_s \right]
\end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{r=1}^n \left[\sum_{i=1}^n m_i \left(\frac{\partial x_i}{\partial q_r} \frac{\partial x_i}{\partial q_t} + \frac{\partial y_i}{\partial q_r} \frac{\partial y_i}{\partial q_t} \right. \right. \\
& \quad \left. \left. + \frac{\partial z_i}{\partial q_r} \frac{\partial z_i}{\partial q_t} \right) \dot{q}_r \right] \\
& + \sum_{i=1}^n m_i \left\{ \left(\frac{\dot{\partial x}_i}{\partial t} \right)^2 + \left(\frac{\dot{\partial y}_i}{\partial t} \right)^2 + \left(\frac{\dot{\partial z}_i}{\partial t} \right)^2 \right\}
\end{aligned}$$

Hence

$$2T = \sum_{r=1}^n \sum_{s=1}^n a_{rs} \dot{q}_r \dot{q}_s + 2 \sum_{r=1}^n a_r \dot{q}_r + a$$

where a_{rs} , a_r , a are called components of inertia.

This is the required expression for kinetic energy of a rigid body .