M.S c Mathematics –SEM 3 Functional Analysis-L-4 CC-11 Unit 1

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Theorem

Let L_∞ denote the set of all bounded sequences $x = (x_i)$ of real or complex numbers. Then L_∞ is a Banach space if for $x = (x_i)$, $y = (y_i) \in l_\infty$ and scalar λ .

We define

$$x + y = (x_i + y_i), \lambda x = (\lambda x_i)$$

And $|| x || = \sup\limits_{i \in N} |x_i|$ **as the norm of** x.

Proof

It is easy to see that l_∞ is a linear space.

Also $|| x || > 0,$

$$|| x || = 0 , iff \; sup \, | x_i | = 0, iff \; | x_i | = 0$$

(for each i) iff $x_i = 0$ (for each i) iff $x = 0$

$$||\lambda x|| = Sup\ |\lambda x_i|$$

$$= |\lambda|\ Sup|x_i|$$

$$= |\lambda|\ \ ||x||$$

Again, we know

$|x_i + y_i| \leq |\ x|\ +\ |\ y\ | \leq ||x||\ +\ ||y||$

So, $||x + y|| = Sup\ |x_i + y_i| \leq ||x|| + ||y||$

Thus $L^{\infty}$is a normed linear space.

We know the metric defined by the norm is given by

$d(x, y) = ||x\ - y||$

$= Sup\ |x_i - y_i|$ **for** $x, y\ \in\ L_{\infty}$

We now show that (l_{∞}, d) is a complete metric space.

Let (x^n) be a Cauchy sequences in l_{∞}.

Given $\in > 0$, there exists $\boldsymbol{n_o} = \boldsymbol{n_o}(\in)$ in N such that

$$\boldsymbol{d}(\boldsymbol{x}^{(n)}, \boldsymbol{x}^{(m)}) < \in \ for\ n, m \geq \boldsymbol{n_0}(\in)$$

Let $\boldsymbol{x}^{(n)} = (\boldsymbol{x}_i^{(n)})$

Then $\boldsymbol{d}(\boldsymbol{x}^{(n)}, \boldsymbol{x}^{(m)}) = \operatorname*{\mathbf{Sup}}_{i} |\boldsymbol{x}_i^{(n)}$-$\boldsymbol{x}^{(m)}| < \in$ for $\boldsymbol{n}, \boldsymbol{m} \geq \boldsymbol{n_0}(\in)$

Hence for each fixed $\boldsymbol{i}$,

$|\, \boldsymbol{x}_i^{(n)}$-$\boldsymbol{x}^{(m)}| < \in$ for $\boldsymbol{n}, \boldsymbol{m} \geq \boldsymbol{n_0}(\in)$..............(i)

Therefore for each fixed i,$(\boldsymbol{x}_i^{(n)})$ is a Cauchy sequence of numbers and hence

$(\boldsymbol{x}_i^{(n)}) \rightarrow \boldsymbol{x_i}$ as $\boldsymbol{n} \rightarrow \infty$e have

Now from (i) $\boldsymbol{m} \rightarrow \infty$

$|\, \boldsymbol{x}_i^{(n)}$-$\boldsymbol{x_i}| \leq \in$ for $\boldsymbol{n}, \geq \boldsymbol{n_0}(\in)$

and each fixed i...............(ii)

$|\boldsymbol{x_i}| = |\boldsymbol{x_i} - \boldsymbol{x}_i^{(n)} + |\, \boldsymbol{x}_i^{(n)}|$

$\leq |x_i - x_i^{(n)}| + |x_i^{(n)}|$

$\leq \in + \underset{i}{\text{Sup}}\, |x_i^{(n)}|$

for $n, \geq n_0(\in)$**.....................by (ii)**

$= \in + ||x^{(n)}||$

Now ,$||x^n|| \leq ||x^{(n)} - x^{(n_o)}|| + ||x^{(n_0)}||$

$< \in + ||x^{(n_0)}||\ n, \geq n_0(\in)$

Now we choose

$M = max\,\{||\,x^{(1)}||, \dots\dots\dots\dots..\,||x^{(n_0-1)}||$ **,**

$\in + ||x^{(n_0)}||$

We have $||x^{(n)}|| \leq M$ **for all** $n \in N$

So $|x_i| \leq \in + M$

Hence $x = (x_i) \in l_\infty$

So from inequality (ii) we have

$$d(x^{(n)}, x) = \underset{i}{\text{Sup}}\, |x_i^{(n)} - x_i| \leq \in$$

for $\boldsymbol{n}, \geq \boldsymbol{n_0}(\in)$

Therefore , $\boldsymbol{x^n} \to \boldsymbol{x} \in \boldsymbol{l}_\infty$

Hence ($\boldsymbol{l}_\infty, \boldsymbol{d}$) is a complete metric space.

Therefore ($\boldsymbol{l}_\infty, \boldsymbol{d}$) is a Banach space.