e-content

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Topic: Inner Product Spaces.

Def.(Inner Product): *Let L(F) be a linear space over the field F (where F is the field of real numbers or the field of complex numbers) then a function (.) from L×L to F*

is called an Inner Product on the linear space L(F)

if it satisfies the following conditions

(1) $(x, x) \geq 0$

(2) $(x, x) = 0 \Leftrightarrow x = 0$

(3) $(x, y) = \overline{(y, x)}$ *[here* $\overline{(y, x)}$ *is the conjugate complex number of* (y, x).

(4) $(ax + by, z) = a(x, z) + b(y, z)$ *for all* $x, y, z \in L$ *and* $a, b \in F$

Inner Product space*: A linear space L with an inner product on it is called an inner product space.*

EXAMPLE: Let $R^2(R)$ *be a linear space where vector addition and scalar multiplication is defined by,*

If $x = (x_1, x_2), y = (y_1, y_2) \in R^2(R)$ *and* $a \in R$,

$$x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$ax = a(x_1, x_2) = (ax_1, ax_2)$$

Now we defined the function (.) from $R^2(R) \times R^2(R)$

to R *by*

$$(x, y) = x_1 . y_1 + x_2 . y_2$$

then (.) is an inner product on $R^2(R)$ *and*

hence $R^2(R)$ *is an inner product Space .*

Verification: $(x, x) = x_1 . x_1 + x_2 . x_2 = x_1^2 + x_2^2 \geq 0$

$$(x, x) = 0 \Leftrightarrow x_1^2 + x_2^2 = 0$$

$$\Leftrightarrow x_1 = 0 \text{ and } x_2 = 0$$

$$\Leftrightarrow x = 0$$

$\overline{(y,x)}$ =$\overline{y_1.x_1+y_2.x_2}$=$y_1.x_1+y_2.x_2$

$= x_1.y_1+x_2.y_2$

$=(x,y)$

$(ax+by,z) = ((ax_1+by_1, ax_2+by_2),(z_1,z_2))$

$=(ax_1+by_1)z_1+(ax_2+by_2)z_2$

$= a(x_1z_1)+b(y_1z_1)+ a(x_2z_2)+b(y_2z_2)$

$= a\{(x_1z_1)+(x_2z_2)\}+b\{(y_1z_1)+(y_2z_2)\}$

$= a(x,z)+b(y,z)$

Hence all the conditions of inner product are satisfied

so the function (.) is an inner product on $R^2(R)$.

And hence $R^2(R)$ *is an inner product space .*

In general

$R^n(R)$ *is an inner product space with respect to the inner product defined by*

$(x,y) = x_1.y_1+x_2.y_2$ +.....+ $x_n.y_n$

$=\sum_{i=1}^{n} x_i\, y_i$

THEOREM: In an inner product space E prove that

(1) $(ax - by, z) = a(x, z) - b(y, z)$

(2) $(x, ay + bz) = \bar{a}(x, y) + \bar{b}(x, z)$

(3) $(x, ay - bz) = \bar{a}(x, y) - \bar{b}(x, z)$

(4) $(0, x) = 0$, $(x, 0) = 0$ *for all* $x \in E$

PROOF: (1) we have

$$(ax - by, z) = (ax + (-b)y, z)$$

$$= a(x, z) + (-b)(y, z)$$

$$= a(x, z) - b(y, z)$$

(2) $(x, ay + bz) = \overline{(ay + bz, x)}$

$$= \overline{a(y, x) + b(z, x)}$$

$$= \bar{a}\overline{(y, x)} + \bar{b}\overline{(z, x)}$$

$$= \bar{a}(x, y) + \bar{b}(x, z).$$

(3) $(x, ay - bz) = (x, ay + (-b)z)$

$$= \bar{a}(x, y) + \overline{(-b)}(x, z)$$

$$= \bar{a}(x, y) + \overline{(-1)b}(x, z)$$

$$= \bar{a}(x, y) + (-1)\bar{b}(x, z).$$

$$= \bar{a}(x, y) - \bar{b}(x, z) .$$

(4) We have $(0.0, x)$ =*0.*$(0, x)$ =*0*

And $(x, 0) = \overline{(0, x)} = \bar{0} = 0$

End