M.S c Mathematics –SEM 2 Functional Analysis-L-3 CC-11 Unit 1

E-content 2–Dr Abhik Singh,

Guest faculty, PG Department of Mathematics, Patna University,

Patna.

Linear Space (Vector Space)

Let $(V,\oplus,\odot)$ be an algebraic structure. then it is called a vector space over a field $(F,+,\bullet)$ if for all $V_1, V_2, V_3 \in V$

$\alpha_1, \alpha_2, \alpha_3 \in F$ **following properties**

(i) **(V, $\oplus$)** **is abelian group.**

(ii) **Distributive for scalar multiplication**

$$\alpha \odot (V_1 \oplus V_2) = \alpha \odot V_1 \oplus \alpha \odot V_2$$

(iii) **Distributive for scalar addition**

$$(\alpha_1 + \alpha_2) \odot V = \alpha_1 \odot V + \alpha_2 \odot V$$

(iv) **Associative for Scalar Multiplication**

$$\alpha \odot (\beta \odot V) = (\alpha \bullet \beta) \odot V$$

(v) $1 \bullet V = V$, **Where 1 is identity element of f with respect to multiplication.**

Then $(V,\oplus,\odot)$ is a Vector Space over F.(linear Space)

Normed Linear Space

Let X be a linear space or vector space over field F and the function , || . ||: $X \to R$ is defined such that

(i) || X ||$\geq$ 0 , || X ||= 0 , $\leftrightarrow$ (if and only if) x = 0 ,$\forall x \in X$

(ii) || x + y || $\leq$|| x || + || y || , $\forall x, y \in X$(Triangular Property)

(iii) || αx | | = |α| || x||,

Then || . || is called norm on X.
And structure $< X, ||\,.\,|| >$ is called Normed Linear space.

NORM→Length of Vector

$d(x-y) = ||x-y||$

$\overrightarrow{BA}$= Position of vector of a – Position Vector of B

Important Theorem

Prove that every Normed linear space is a Metric space but converse may not be true.

Proof

To prove this we know in metric space if X is a arbitrary set then X

d: X x X $\longrightarrow$ R which satisfied the following condition

(i) $d(x,y) \geq 0$

(ii) $d(x,y)=0$ if and only if $x=y$

(iii) $d(x,y)=d(y,x)$

(iv) $d(x,y) \leq d(x,z)+ d(z,y)$

Let X be a normed linear space , then

we have

$d(x,y) = || x - y||, \forall x, y \in X$

(i) $d(x, y) = || x - y ||, \forall x, y \in X$

(ii) $d(x, y) = 0$ (if and only if), $\leftrightarrow || x - y|| = 0$

$\leftrightarrow x-y=0$

$\leftrightarrow x=y$

(iii) $d(x, y) = || x- y ||$

$= ||-(y-x) ||$

$= |-1| \; || y-x||$

$= ||y-x|| = d(y,x)$

(iv) $d(x, y) = || x - y ||$

$= || (x-z) + (z - y) ||$

$\leq || x - Z || + || z - y ||$

$\leq d(x, z) + d(z, y)$

$\textit{that implies } d(x,y) \leq d(x,z) + d(z,y)$

Hence every normed linear space is a metric space.

Proved

Converse may not be true

e .g

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

$|| x - y || = \frac{|x-y|}{1+|x-y|}$, put $Z = x-y \in X$

$$|| Z || = \frac{|Z|}{1+|Z|}$$

$$|| \alpha Z || = \frac{|\alpha Z|}{1+|\alpha Z|} = \frac{|\alpha|\ |Z|}{1+|\alpha|\ |Z|} = |\alpha|\left(\frac{|Z|}{1+|\alpha|\ |Z|}\right)$$

$$\neq |\alpha|\ ||Z||$$

So , X is not normed liner space