Equation of continuity in Co-ordinate system (3)

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1 The equation of continuity in Cartesian coordinates.

Let there be a fluid particle at P(x, y, z). Let $\rho(x, y, z, t)$ be the density of the fluid at P at ant time tand let u, v, w be the velocity components at P parallel to the rectangular coordinate axes. Construct a small parallel piped with edges $\partial x, \partial y, \partial z$ of lengths parallel to their respective shown in figure. Then, we have

Mass of the fluid that passes in through the face $PQRS = (\rho \partial y \partial z)u$ per unit time = f(x, y, z)(say)

(1)

 \therefore Mass of the fluid that passes out through the opposite face P'Q'R'S'

$$= f(x + \partial x, y, z) \text{per unit time} = f(x, y, z) + \delta x \frac{\partial}{\partial x} f(x, y, z) + \dots$$
(2)

 \therefore The net gain in mass per unit time within the element (rectangular parallelepiped due to flow through the faces PQRS and P'Q'R'S' by using (1) and (2).

= Mass that enters in through the face PQRS - Mass that leaves through the face P'Q'R'S'

 $= f(x, y, z, w) - \left[f(x, y, z, w) + \delta \cdot \frac{\partial}{\partial x} f(x, y, z) + \dots \right]$

 $=-\delta x \cdot \frac{\partial}{\partial x} f(x, y, z)$, to the first order of approximation $=-\delta x \cdot \frac{\partial}{\partial x} (\rho u \delta y \delta z)$ by equation (1)

$$= -\delta x \delta y \delta z \frac{\partial(\rho u)}{\partial x} \tag{3}$$

similarly, the net gain in mass per unit time within the element due to flow through the faces PP'S'S and QQ'RR'

$$=\delta x \delta y \delta z \frac{\partial(\rho v)}{\partial y} \tag{4}$$

and the net again in mass per unit time within the element sue to flow through the faces PP'Q'Q and SS'RR'

$$=\delta x \delta y \delta z \frac{\partial(\rho w)}{\partial z} \tag{5}$$

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Figure 1: Schematic of the fluid flow in cartesian form.

... The total rate of mass flow into the elementary parallelpiped

$$= -\delta x, \delta y, \delta z \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right]$$
(6)

Again, the mass of the fluid within the element at time $t = \rho \delta x \delta y \delta z$

 \therefore Total rate of mass increase within the element

$$= \frac{\partial}{\partial t} (\rho \delta x \delta y \delta x) = \delta x \delta y \delta x \frac{\partial \rho}{\partial t}$$
(7)

Suppose that the chosen region (bounded by the elementary parallelepiped) of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must he equal to the rate of mass flowing into the element. Hence from (6) and (7), we have

$$\delta x \delta y \delta x \frac{\partial \rho}{\partial t} = -\delta x \delta y \delta z \left[\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right]$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{8}$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial w}{\partial z} + \rho \frac{\partial \rho}{\partial z} = 0$$

$$\Rightarrow \quad \left[\frac{\partial}{\partial x} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial x} + w \frac{\partial}{\partial x} \right] \rho + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

$$\Rightarrow \quad \frac{D\rho}{Dt} + \rho \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} \right) \tag{9}$$

Which is the desire equation of continuity in Cartesian coordinates and it holds at all point of the fluid free sources and sinks.

Remark. If the fluid is heterogeneous and incompresible $.\rho$ is a constant and (9) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

Further, if the fluid is heterogeneous and in-compressible ρ is function of x, y, z and t such that $\frac{D\rho}{Dt} = 0$. Hence the corresponding equation of the continuity is again given by (10).

2 Equation of continuity in different coordinate

A fluid density (ρ) flow with velocity q. Then

	Cartesian			Cylindrical			Spherical		
axes	x	y	z	r	θ	z	r	θ	ϕ
Velocity(\boldsymbol{q})	u	v	w	q_v	q_{θ}	q_z	q_r	q_{θ}	q_{ϕ}
Changes	δx	δy	δz	δr	$\delta \theta$	δz	δv	$r\delta\theta$	$r\sin\theta\delta\phi$

1. Cartesian Co-ordinate (x, y, z) system

$$\frac{\partial(\rho\delta x\delta y\delta x)}{\partial t} + \delta x \frac{\partial(\rho u\delta y\delta z)}{\partial x} + \delta y \frac{\partial(\rho v\delta x\delta z)}{\partial y} + \delta z \frac{\partial(\rho w\delta y\delta x)}{\partial z} = 0$$
$$\implies \quad \frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

which is fluid continuity equation in Cartesian form.

2. Cylindrical Co-ordinate (r, θ, z) system

$$\frac{\partial(\rho r \delta r \delta \theta \delta z)}{\partial t} + \delta r \frac{\partial(\rho q_r \delta \theta r \delta z)}{\partial r} + \delta \theta \frac{\partial(\rho q_\theta \delta r \delta z)}{\partial \theta} + \delta z \frac{\partial(\rho q_z r \delta r \delta \theta)}{\partial z} = 0$$
$$\implies \quad \frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho r q_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho q_\theta)}{\partial \theta} + \frac{\partial(\rho q_z)}{\partial z} = 0$$

3. Spherical Co-ordinate (r, θ, ϕ) system

$$\frac{\partial(\rho\delta r\delta\theta r^2\sin\theta\delta\phi)}{\partial t} + \delta r \frac{\partial(\rho q_r \delta\theta r^2\sin\theta\delta\phi)}{\partial r} + \delta \theta \frac{\partial(\rho q_\theta r\sin\theta\delta r\delta\phi)}{\partial \theta} + \delta \phi \frac{\partial(\rho q_\phi r\delta r\delta\theta)}{\partial \phi} = 0$$
$$\implies \quad \frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho r^2 q_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho q_\theta)}{\partial \theta} + \frac{1}{r\sin\theta} \frac{\partial(\rho q_\phi)}{\partial \phi} = 0$$

All the best...

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