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SEMESTER – IV

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Topic : Weierstrass Approximation Theorem

Topic - 1

Weierstrass Approximation Theorem

Before giving this theorem we prove same facts :-

For every natural number n and $x \in [0,1]$

We have

(i) $$\sum_{r=0}^{n} P_{nr}(x) = \sum_{r=0}^{n} \mathrm{n}_{\mathrm{c}_\mathrm{r}} x^{\mathrm{r}} (1-x)^{n-r} = 1$$

(ii) $$\sum_{r=0}^{n} r P_{nr}(x) = \sum_{r=1}^{n} \mathrm{r}.\, \mathrm{n}_{\mathrm{c}_\mathrm{r}} x^{\mathrm{r}} (1-x)^{n-r} = nx$$

(iii) $$\sum_{r=0}^{n} (nx-r)^2 P_{nr}(x) = nx(1-x)$$

Proof (i) we have, by the binomial theorem

$$\sum_{r=0}^{n} P_{nr}(x) = \sum_{r=0}^{n} \mathrm{n}_{\mathrm{c}_\mathrm{r}} x^{\mathrm{r}} (1-x)^{n-r}$$

$$= [x + (1-x)]^n$$

$$= (1)^n = 1.$$

Proof (ii) : Since $r.n C_r = r.\frac{\lfloor n}{\lfloor r.\lfloor n-r}$

$$= r.\frac{n.\lfloor n-1}{r.\lfloor r-1.\lfloor n-r}$$

$$= n.\frac{\lfloor n-1}{\lfloor r-1 \lfloor (n-1)-(r-1)}$$

$$= n.n-1_{C_{r-1}}$$

So $$\sum_{r=0}^{n} r.P_{nr}(x) = \sum_{r=1}^{n} r.n_{C_r} x^{\mathrm{r}} (1-x)^{n-r}$$

$$= \sum_{r=1}^{n} n.x.\ \ n-1_{C_{r-1}}\ x^{r-1}(1-x)^{n-r}$$

$$= n\,x \sum_{r=1}^{n}\ n-1_{C_{r-1}}\ x^{r-1}(1-x)^{n-r}$$

Let $r-1 = S \Rightarrow r = 1+S$

if $r = 1 \Rightarrow S = 0$

if $r = n \Rightarrow S = n-1$

So, from (2) we get

$$\sum_{r=0}^{n} r.P_{nr}(x) = nx.\sum_{s=0}^{n-1} n-1_{C_s} x^{\mathrm{s}}(1-x)^{(n-1)-s}$$

$$= nx.[x+(1-x)]^{n-1} = nx.(1)^{n-1}$$

$$= nx$$

Proof (iii) Since

$$= \sum_{r=2}^{n} r.(r-1)\ n_{C_r} \quad x^{\mathrm{r}}(1-x)^{n-r}$$

$$= \sum_{r=2}^{n} n.(n-1) \quad n-2_{C_{r-2}} \quad x^{\mathrm{r}}(1-x)^{n-r}$$

$$\left[\because\ r.(r-1)n_{C_r} = n(n-1)n-2_{C_{r-2}}\right]$$

$$= n.(n-1)x^2\ .\ \sum_{r=2}^{n} n-2_{C_{r-2}} \quad x^{\mathrm{r}-2}(1-x)^{n-r}$$

$$= n.(n-1)x^2\ .\ \sum_{s=0}^{n-2} n-2_{C_S} \quad x^{\mathrm{S}}(1-x)^{(n-2)-s} \qquad [\text{Taking}\, r-2=s]$$

$$= n(n-1)x^2[x+(1-x)]^{n-2}$$

$$= n(n-1)\ x^2\ (1)^{n-2} = n(n-1)x^2$$

Now, $\sum_{r=0}^{n}(nx-r)^2 P_{n\,r}(x)$

$$= \sum_{r=0}^{n} n^2x^2P_{nr}(x) - 2nx\sum_{r=0}^{n} r\,P_{nr}(x) + \sum_{r=0}^{n} r^2P_{nr}(x)$$

$$= \sum_{r=0}^{n} n^2x^2P_{nr}(x) - 2nx\sum_{r=0}^{n} r\,P_{nr}(x) + \sum_{r=0}^{n} r(r-1)P_{nr}(x) + \sum_{r=0}^{n} rP_{nr}(x)$$

$$= n^2x^2 - 2nx.nx + n(n-1)x^2 + nx$$

$$= n^2x^2 - 2n^2x^2 + n^2x^2 - nx^2 + nx$$

$$= nx(1-x)$$

Def^n : Bernstein Polynomial

Let $f: [0,1] \to R$ be a function, then the polynomial $= B_n(f.x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) P_{nr}(x)$,

where n = 1, 2, …… is called a Bernstein polynomial for f

.

Weierstrass Approximation Theorem:

Let f be a continuous function defined on $[a,b]$. Then there exists a sequence of polynomials which converges uniformly to f on $[a,b]$.

Proof : We prove that theorem for the interval [0, 1]

We shall show that the Bernstein polynomial sequence $\{B_n\}$ is such a sequence which converges uniformly to f on [0, 1].

We have polynomial $B_n(f,x)$ on [0, 1]

$$B_n(f,x) = \sum_{n=0}^{n} f\left(\frac{r}{n}\right) P_{n\,r}(x)$$

where $P_{nr}(x) = n_{C_r} x^r (1-x)^{n-r}$

Let $\in > 0$ be given

Then $\frac{\in}{2} > 0$ and f is continuous on [0, 1], So it is uniformly continuous on [0, 1]

Hence for $\frac{\in}{2} > 0 \quad \exists \ \delta > 0$

such that $x, y \in [0,1]$ with $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\in}{2}$ .... (i)

Again f is continuous on [0,1] so it is bounded on [0,1]. Hence there exists a positive real number M such that

$$|f(x)| \leq M \text{ for all } x \in [0,1] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2)$$

Now for every n = 1, 2, ……. and $x \in [0,1]$,

$$|f(x) - B_n(f,x)| = \left| f(x) - \sum_{r=0}^{n} f\left(\frac{r}{n}\right) P_{nr}(x) \right|$$

[from fact (i)]

$$= \left| \sum_{r=0}^{n} \left[f(x) - f\left(\frac{r}{n}\right) \right] P_{nr}(x) \right|$$

$$= \sum_{r=0}^{n} \left| f(x) - f\left(\frac{r}{n}\right) \right| . P_{nr}(x)$$

$$= \sum_{\left|x - \frac{r}{n}\right| < \delta} \left| f(x) - f\left(\frac{r}{n}\right) \right| . P_{nr}(x) + \sum_{\left|x - \frac{r}{n}\right| \geq \delta} \left| f(x) - f\left(\frac{r}{n}\right) P_{nr}(x) \right|$$

$$\leq \sum_{\left|x - \frac{r}{n}\right| < \delta} \left| f(x) - f\left(\frac{r}{n}\right) \right| . P_{nr}(x) + \sum_{\left|x - \frac{r}{n}\right| \geq \delta} |f(x)| P_{nr}(x) + \sum_{\left|x - \frac{r}{n}\right| \geq \delta} \left| f\left(\frac{r}{n}\right) \right| P_{nr}(x)$$

$$< \frac{\epsilon}{2} + M \sum_{\left|x - \frac{r}{n}\right| \geq \delta} P_{nr}(x) + M \sum_{\left|x - \frac{r}{n}\right| \geq s} P_{nr}(x)$$

[from (1) & (2)]

$$= \frac{\in}{2} + 2M \sum_{\left|x-\frac{r}{n}\right|\geq\delta} P_{nr}(x)$$

$$\leq \frac{\in}{2} + 2M \sum_{\left|x-\frac{r}{n}\right|\geq\delta} \frac{(nx-r)^2}{n^2\delta^2} P_{nr}(x) \qquad \left[\because \left|x - \frac{r}{n}\right| \geq \delta \Rightarrow (nx-1)^2 \geq n^2\delta^2\right]$$

$$= \frac{\in}{2} + \frac{2M}{n^2\delta^2} \sum_{r=0}^{n} (nx-r)^2 P_{nr}(x)$$

$$= \frac{\in}{2} + \frac{2M}{n^2\delta^2} . nx \ (1-x) \qquad \text{[from fact (iii)]}$$

$$= \frac{\in}{2} + \frac{2M}{n^2\delta^2} . n . \frac{1}{4} \qquad \left[\because \ x(1-x) \leq \frac{1}{4} \forall \ x \in [0,1]\right]$$

$$= \frac{\in}{2} + \frac{M}{2\,n\,\delta^2} . n . \frac{1}{4} \quad \text{..........................(3)}$$

Now, Since M, $\in, \delta^2$ all are positive, So $\frac{M}{\in\delta^2}$ is also positive real number. Hence by Archimedean property of real number, there exists a positive integer m such that $\frac{M}{\in\delta^2} < m$

if $n \geq m \quad \Rightarrow \frac{1}{n} \leq \frac{1}{m}$

$$\Rightarrow \frac{M}{n\delta^2} \leq \frac{M}{m\delta^2} < \in$$

$$\Rightarrow \frac{M}{2\,n\,\delta^2} < \frac{\epsilon}{2}$$

Hence from (4) if $n \geq m$ then

$$|f(x) - B_n(f, x)| < \frac{\in}{2} + \frac{\in}{2} = \in \quad \forall\, x \in [0, 1]$$

Hence $\{B_n\}$ converges uniformly to f on [0,1]

If f is continuous function on $[a, b]$, then we define a function g on [0, 1] by

$$g(x^{'}) = f[(b-a)x^{'} + a], \qquad x^{'} \in [0, 1]$$

Then g is continuous on [0,1] so there exists a sequence of polynomials $\{B_n\}$ defined over [0,1] such that $\lim_{n\to\infty} B_n^{'}(g, x^{'}) = g(x^{'})$ uniformly on [0,1]

Taking $(b-a)x^{'} + a = x \Rightarrow x \in [a, b]$

and $\lim_{n\to\infty} B_n^{'}\left(\frac{x-a}{b-a}\right) = f(x)$ uniformly on $[a, b]$. If we define for each n

$B_n(x) = B_n^{'}\left(\frac{x-a}{b-a}\right)$ we find that

$$\lim_{n\to\infty} B_n(x) = f(x)$$

uniformly on $[a, b]$. It proves.