Material derivatives and Euler equation of continuity(2)

Binod Kumar[*]
M.Sc. Mathematics ***Semester:*** *III*
Paper: Fluid Dynamics *XII* *(**MAT CC**-12)*
Patna University ,Patna

August 12, 2020

1 Material derivatives

Suppose a fluid particle moves from $p(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Further, suppose $g(x, y, z)$ be a scalar function associate with some physical property. Rate of change of $g(x, y, z)$ be

$$\frac{dg}{dt} = \frac{\partial g}{\partial t}\frac{dx}{dt} + \frac{\partial g}{\partial t}\frac{dy}{dt} + \frac{\partial g}{\partial t}\frac{dz}{dt} \tag{1}$$

$$\text{Say} \quad u = dx/dt, \quad v = dy/dt, \quad w = dz/dt$$

$$\frac{dg}{dt} = u\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial t} + w\frac{\partial g}{\partial t} \tag{2}$$

but

$$\boldsymbol{q} = ui + vj + wk$$

and

$$\boldsymbol{q} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Equation (2) can be written as

$$\frac{Dg}{Dt}(=\frac{dg}{dt}) = \frac{\partial g}{\partial t} + (\mathbf{q} \cdot \nabla)g \tag{3}$$

where $\mathbf{q}$ (u, v, w) is the velocity and

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \tag{4}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{q} \cdot \nabla) \tag{5}$$

The term $\frac{D}{Dt}$ is known material derivative ,$\frac{\partial}{\partial t}$ is the local derivative and it is associated with time variation at fixed position and namely $(\mathbf{q} \cdot \nabla)$ is called the convective derivative,which is associated with

[*]Corresponding author, e-mail:binodkumararyan@gmail.com, Telephone: +91-9304524851

the change of physical quantity due to flow of fluids parcel. Equation (5) is the basic equation to describe fluid mechanics. If **a** is acceleration of fluid,which given by operator $\frac{D\mathbf{q}}{Dt}$ as

(i) Acceleration **a** of fluid particle in Cartesian's coordinate system (x, y, z) be

$$\frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} \tag{6}$$

Acceleration $\mathbf{a} = (a_x, a_y, a_z)$ of fluid particle in cartesian- coordinate system (x, y, z) be

$$\begin{cases} a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{cases} \tag{7}$$

(ii) Acceleration **a** of fluid particle in cylindrical coordinate system (r, θ, z) be

$$\begin{cases} a_1 = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} \\ a_2 = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial w}{\partial z} + \frac{uv}{r} \\ a_3 = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} \end{cases} \tag{8}$$

(iii) Component of acceleration **a** (=(a_1, a_2, a_3)) of the fluids particle in spherical polar coordinate system (r, θ, ϕ) be

$$\begin{cases} a_1 = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{v^2+w^2}{r} \\ a_2 = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \phi} + \frac{uv}{r} - \frac{w^2\cot\theta}{r} \\ a_3 = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{uw}{r} + \frac{wv\cot\theta}{r} \end{cases} \tag{9}$$

the component of acceleration along the axes in different coordinate system.

1. **Example:** If the velocity distribution $\boldsymbol{q} = iAx^2y + jBy^2zt + kzt^2$, Where A, B and C are constants the find theacceleration and velocity components.
   Solution:-

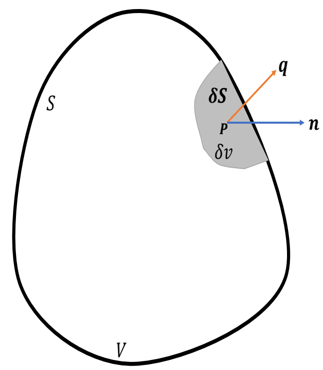
$$a = \left[A(2Ax^3y^2 + Bx^2y^2zt)i + B(y^2z + 2By^3z^2t^2 + Cy^2zt^3)j + C(2zt + Czt^4)k\right]$$

2 Conservation of Mass:

The law of conservation of mass state that fluid mass can't be either created or destroyed. It can be transfer. So our need to expression to mathematical form of model of mass conservation. so sake simplicity conservation of mass in Euler system.

2.1 The equation of continuity,(or equation of conservation of mass) by Euler's method.

Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let $P(x, y, z)$ be any point of S and let $\rho(x, y, z, t)$ be the fluid density at P any time t. Let ∇S denote element of the surface S enclosing P. Let n be the unit outward-drawn



normal at ∇S and let q be the fluid velocity at P. Then the normal component of $\boldsymbol{q}$ measures outwards from V is $n \cdot \boldsymbol{q}$. Thus,

Rate of mass flow across $\quad \delta S = \rho(n \cdot \boldsymbol{q})\delta S$

$\therefore$ Total rate of mass flow across S

$$\int\limits_{s} \rho(n \cdot \boldsymbol{q})\delta S = \int\limits_{V} \nabla \cdot (\rho \boldsymbol{q}) dV$$

(By Gauss divergence theorem)

$\therefore$ Total flow of mass flow into

$$V = -\int\limits_{V} \nabla \cdot (\rho \boldsymbol{q}) dV$$

Again, the mass of the fluid within S at time

$$t = -\int\limits_{V} \rho \, dV$$

$\therefore$ Total rate of mass increase within

$$S = \frac{\partial}{\partial t} \int\limits_{V} \rho dV = \int\limits_{V} \frac{\partial \rho}{\partial t} dV$$

Suppose that the region V of the fluid contains neither sources nor sinks (i.e. there no inlets or outlets through which fluids can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from from above equation, We have

$$\int\limits_{V} \frac{\partial \rho}{\partial t} dV = -\int\limits_{V} \nabla \cdot (\rho \boldsymbol{q}) dV \Rightarrow \int\limits_{V} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{q}) \right] dV$$

Which holds for arbitrary small volumes V, if

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{q}) = 0$$

This equation is called equation of continuity, or conservation of mass and it holds as all points of fluid free from sources and sinks.

1. Since $\nabla \cdot (\rho \boldsymbol{q}) = \rho \nabla \cdot \boldsymbol{q} + \nabla \rho \cdot \boldsymbol{q}$, therefore other form of above equation is
$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \boldsymbol{q} + \nabla \rho \cdot \boldsymbol{q} = 0$$
$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{q} = 0$$
$$\frac{D(\log \rho)}{Dt} + \nabla \cdot \boldsymbol{q} = 0$$

2. For an in-compressible and hetrogeneous fluid the density of any fluis particle is invariable with time so that $\frac{D\rho}{Dt} = 0$ then,
$\nabla \cdot \boldsymbol{q} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ if $\boldsymbol{q} = ui + vj + wk$.

3. For an in-compressible and homogeneous fluid, ρ is constant and hence $\frac{\partial \rho}{\partial t} = 0$
$\nabla \cdot (\rho \boldsymbol{q}) = 0 \Rightarrow \nabla \cdot \boldsymbol{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, as ρ is constant.

Next in 3^{rd} Econtent

All the best...