LPP (Simplex Method)
(M.Sc. Sem-III)

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Use simplex method to :

Minimize $z = x_2 - 3x_3 + 2x_5$ subject to the constraints :

$$3x_2 - x_3 + 2x_2 \le 7,\ -2x_2 + 4x_3 \le 12,$$

$$-4x_2 + 3x_3 + 8x_5 \le 10;\ x_2 \ge 0,\ x_3 \ge 0 \text{ and } x_5 \ge 0.$$

Solution : Introducing slack variables $s_1 \ge 0,\ s_2 \ge 0$ and $s_3 \ge 0$ in the respective inequalities; and converting the objective function into that of maximization; the linear programming problem is :

Maximize $z^* = -(x_2 - 3x_3 + 2x_5) + 0.s_1 + 0.s_2 + 0.s_3$ subject to the consraints :

$$3x_2 - x_3 + 2x_5 + s_1 = 7$$

$$-2x_2 + 4x_3 + 0.x_5 + s_2 = 12$$

$$-4x_2 + 3x_3 + 8x_5 + s_3 = 10$$

$$x_2,\ x_3,\ x_5,\ s_1,\ s_2,\ s_3 \ge 0.$$

The iterative simplex tables are :

Initial iteration. Introduce y_2 and drop y_5.

			-1	3	-2	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	7	3	-1	2	1	0	0
0	y_5	12	-2	**4**	0	0	1	0
0	y_6	10	-4	3	8	0	0	1
	z	0	1	-3	2	0	0	0

At least one $z_j - c_j$, *viz*, $z_2 - c_2$ is negative and therefore the current basic feasible solution is optimum. We choose the column corresponding to $z_2 - c_2$, i.e., column vector y_2 enters the basis (since at least one $y_{i2} > 0$). Further, since minimum $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\} \frac{12}{4}(=3)$, current basis vector y_5 leaves the basis. This gives $y_{22}(=4)$ as the leading element. Now using E-row operations, we convert the leading element into unity and all other elements of the entering column vector y_2 to zero. We get the improved basic feasible solution as show in the next simplex table.

First Iteration. Introduce y_1 and drop y_4.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	10	5/2	0	2	1	$1/4$	0
3	y_2	3	$-1/2$	1	0	0	$1/4$	0
0	y_6	1	$-5/2$	0	8	0	$-3/4$	1
	z	9	$-1/2$	0	2	0	$3/4$	0

Observe that $z_1 - c_1$ is negative and thus the current basic feasible solution is not optimum. The column vector corresponding to $z_1 - c_1$ enters the next basis $y_B\left(\text{since } y_{i1} > 0\right)$. Further, since only $y_{11} > 0$ both $y_{12} < 0$ and $y_{13} < 0$); current basis vector y_4 leaves the basis. This gives $y_{11}\left(=\frac{5}{2}\right)$ as the leading element. Using E-Row operations, we convert the leading element into unity and all other entries in its column y_1 to zero. The improved basic feasible solutions is obtained as shown in the next simplex table.

Final Iteration. Optimum Solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
-1	y_1	4	1	0	$4/5$	$2/5$	$1/10$	0
3	y_2	5	0	1	$2/5$	$1/5$	$3/10$	10
0	y_6	11	0	0	10	$2/5$	$-1/2$	1
	z	11	0	0	$12/5$	$1/5$	$8/10$	0

Since all $z_j - c_j \geq 0,$ an optimal basic feasible solution has been attained. Thus, the optimum solution to the given L.P.P. is

$$\text{Minimum } z = -\text{ Maximum } z^* = -11 \text{ with } x_2 = 4,\ x_3 = 5 \text{ and } x_5 = 0$$

1. Two-Phase Method; and
2. Big-M Method or Method of Penalties.

TWO-PHASE METHOD

In the first phase of this method, the sum of the artificial variables is minimized subject to the given constraints (known as auxiliary L.P.P.) to get a bsic feasible solution to the original L.P.P. Second phase then optimizes the original objective function starting with the basic feasible solution obtained at the end of Phase 1.

The iterative procedure of the algorithm may be summarise as below :

Step-1 : Write the given L.P.P. into its standard form and check whether there exists a starting basic feasible solution.

(a) If there is a ready starting basic feasible solution, go to Phase-2.

(b) If there does not exist a ready starting basic feasible solution, go on to the next step.

PHASE-I

Step-2 : Add the artificial variable to the left side of each equation that lacks the needed starting basic variables. Construct an auxialiary objective function aimed at minimising the sum of all artificial variables.

Thus, the new objective is to

$$\text{Minimize } z = A_1 + A_2 + \ldots + A_n$$

$$\text{Maximize } z^* = -A_1 - A_2 - \ldots - A_n$$

where $A_i\left(i = 1,\ 2,\ \ldots,\ m\right)$ are the non-negative artificial variables.

Step-3 : Apply simplex algorithm to the specially constructed L.P.P. The following three cases may arise at the least interaction :

(a) max $z^* < 0$ and at least one artificial variable is present in the basis with positive value. In such a case, the original L.P.P. does not possess any feasible solution.

(b) max $z^* = 0$ and at least one artificial variable is present in the basis at zero value. In such a case, the original L.P.P. possess the feasible solution. In order to get basic feasible solution we may proceed directly to Phase-2 or else eliminate the artificial basic variable and then proceed to Phase-2.

(c) maximum $z^* = 0$ and no artificial variable is present in the basis. In such a case, a bsic feasible solution to the original L.P.P. has been found. Go to Phase-2.

PHASE-2

Step-4 : Consider the optimum basic feasible solution of Phase-1 as a starting basic feasible solution for the original L.P.P. Assign actual coefficients to the variables in the objective function and a value zero to the artificial variables that appear at zero value in the final simplex table of Phase-1.

Apply usual simplex algorithm to the modified simplex table to get the optimum solution of the original problem.

Note : Artificial variables that do not appear in the basic solution may be deleted from the simplex table totally.

Use two-phase simplex method to maximize $z = 5x_1 + 3x_2$ subject to the constraints :

$$2x_1 + x_2 \le 1,\ x_1 + 4x_2 \ge 6 \text{ and } x_1,\ x_2 \ge 0.$$

Solution : Introducing a slack variable $s_1 \ge 0$, a surplus variable $s_2 \ge 0$ and an artificial variable $A_1 \ge 0$ in the constraints of the L.P.P., an initial basic feasible solution is : $s_1 = 1$ and $A_1 = 6$ where I_2 as the basis matrix.

Phase-1 : The objective function of the auxiliary L.P.P. is to maximize $z^* = -A_1$.

Using now simplex algorithm to the auxiliary linear programming problem, the iterative simplex tables are:

Initial Iteration. Introduce y_2 and drop y_3.

			0	0	0	0	−1
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_3	1	2	**1**	1	0	0
−1	y_5	6	1	4	0	−1	1
	z_j	–	−1	−4	0	1	−1
	$z_j - c_j$	$z(=-6)$	−1	−4	0	1	0

Since $z_1 - c_1$ and $z_2 - c_2$ are negative, we choose the most negative of these, *viz.*, –4. The corresponding column vector y_2 enters the basis. Further, since min. $\left\{\frac{x_{Bi}}{y_{i2}},\ y_{i2} > 0\right\}$ is 1, which occurs for element y_{12}, y_3 leaves the basis.

Final Iteration. Optimum solution.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
0	y_2	1	2	1	1	0	0
−1	y_5	2	−7	0	−4	−1	1
		$z(=-2)$	7	0	4	1	0

Since all $(z_j - c_j) \ge 0$, an optimum basic feasible solution to the auxiliary L.P.P. is obtained. But max. $z^* < 0$ and an artificial variable is in the basis at a positive level. Therefore, the original L.P.P. does not possess any feasible solution.