LPP (Simplex Method)
(M.Sc. Sem-III)

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Theorem-1 (Extreme Point Correspondence). A basic feasible solution to an L.P.P. must correspond to an extreme point of the set of all feasible solutions and conversely.

Proof. Let the L.P.P. be :

Maximize $z = c^T x,\ x \in R^n$ subject to the constraints :

$$Ax = b \ \ x \geq 0$$

where **A, b** and **c** are real $m \times n$, $m \times 1$ and $n \times 1$ matrices respectively. Let $\rho(A) = m$.

Let *S* be the set of all feasible solutions to the L.P.P. Also suppose that **x** is a basic feasible solution

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix}$$

where x_B is an $m \times 1$ vector, such that for a non-singular sub-matrix **B** of **A**, $Bx_B = b$.

If possible, let **x** be a point of *S*, such that there exist $x_1, x_2 \in S$ such that $x_1 \neq x_2$ and

$$x = \lambda x_1 + (1-\lambda) x_2 \qquad 0 < \lambda < 1$$

Let $$x_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

where u_1, u_2 are $m \times 1$ vectors and v_1, v_2 are $(n-m) \times 1$ vectors. Then

$$\begin{bmatrix} x_B \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + (1-\lambda) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$\therefore$ $$x_B = \lambda u_1 + (1-\lambda) u_2$$

and $$0 = \lambda v_1 + (1-\lambda) v_2 \qquad 0 < \lambda < 1.$$

Since x_1, x_2 are feasible solutions, therefore $u_1, u_2, v_1, v_2 \geq 0$

Now $0 < \lambda < 1$ and $0 = \lambda v_1 + (1-\lambda) v_2$

$\therefore$ We must have $v_1 = v_2 = 0$

Thus, $$x_1 = \begin{bmatrix} u_1 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} u_2 \\ 0 \end{bmatrix}$$

Again, since x_1, x_2 satisfy $Ax_1 = b$ and $Ax_2 = b$, therefore

$$Bu_1 = b \text{ and } Bu_2 = b.$$

Also, since $Bx_B = b$ and since expression of *b* as a linear combination of basis vectors must be unique, therefore, $u_1 = u_2 = x_B$. Hence

$$x = x_1 = x_2.$$

This is a contradiction, for by assumption $x_1 \neq x_2$.

Hence *x* is an extreme point of *S*.

Conversely. We prove that an extreme point $x^* = [x_1, x_2, ..., x_n]$ of the set of all feasible solutions is a basic feasible solution. That is, the column vectors of A associated with non-zero variables are linearly independent.

Since some components (variables) of x^* may be zero, without any loss of generality let us suppose that the first p components of x^* are positive and the remaining $(n-p)$ are zero. Then $Ax^* = b$ implies that

$$x_1 a_1 + x_2 a_2 + ... + x_p a_p = b,$$

where $a_i \in A$.

If possible, let us assume that the vectors $a_1, a_2, ..., a_p$ are not linearly independent. Then, there must exist some scalars α_j not all zero, (tht is at least one $\alpha_j \neq 0$) such that

$$\alpha_1 a_1 + \alpha_2 a_2 + ... + \alpha_p a_p = 0 \text{ or } \sum_{j=1}^{p} \alpha_j a_j = 0.$$

Therefore, for some arbitrary $\varepsilon > 0$, we can write

$$\sum_{j=1}^{p} x_j a_j \pm \varepsilon \sum_{j=1}^{p} \alpha_j a_j = b.$$

or $$\sum_{j=1}^{p} (x_j \pm \varepsilon \alpha_j) a_j = b \qquad (\text{some } \alpha_j \neq 0)$$

Thus, the two different points

$$x^{(1)} = (x_1 + \varepsilon\alpha_1, x_2 + \varepsilon\alpha_2, ..., x_p + \varepsilon\alpha_p, 0, 0, ..., 0)$$

and $$x^{(2)} = (x_1 - \varepsilon\alpha_1, x_2 - \varepsilon\alpha_2, ..., x_p - \varepsilon\alpha_p, 0, 0, ..., 0)$$

satisfy the constraints $Ax^* = b$

Also, since $x_j > 0$, by selecting ε such that

$$0 < \varepsilon < \min\left\{\frac{x_j}{|\alpha_j|}\right\} \qquad j = 1, 2, ..., p$$

the first p components of $x^{(1)}$ and $x^{(2)}$ will always be positive. As the remaining $(n-p)$ components in $x^{(1)}$ and $x^{(2)}$ are zero, it follows that $x^{(1)}$ and $x^{(2)}$ are feasible solutions different from x^*. We observe that

$$\frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)} = (x_1, x_2, ..., x_n) = x^*.$$

This shows that x^* can be expressed as a convex combination of two distinct feasible solutions $x^{(1)}$ and $x^{(2)}$ by selecting $\lambda = \frac{1}{2}$, which contradicts the assumption that x^* is an extreme point.

Consequently, for x^* to be an extreme point, the vectors $a_1, a_2, ..., a_p$ must be linearly independent and hence x^* is a basic solution.

Now, since there are m equations in the system, there can be at most m linearly independent vectors in R^m. Therefore, if $p = m$ (a non-degenerate extreme point), then by definition, there is unique basic feasible solution corresponding to the extreme point. On the other hand, if $p < m$ degenerate extreme point), then one can select $(m-p)$ additional vectors from A with their corresponding variables equal to zero, such that the resulting set of vectors are linearly independent. This completes the proof.

Theorem-2 (Fundamental Theorem of Linear Programming). If the feasible region of an L.P.P. is a convex polyhedron, then there exists an optimal solution to the L.P.P. and at least one basic feasible solution must be optimal.

Proof. Let the L.P.P. be to determine x so as to

Maximize $z = c^T x$ subject to : $Ax = b$ and $x \geq 0$,

where $c,\ x \in R^n$.

Then the feasible region S of the L.P.P. is given by

$$S = \{x \mid Ax = b,\ x \geq 0\}.$$

Since S is a convex polyhedron, it is non-empty, closed and bounded.

The objective function $dz = c^T x,\ x \in S$ is continuous on S, which is non-empty, closed and bounded, therefore, z attains its maximum on S. This proves the existence of an optimal solution.

Now, since S is a convex polyhedron, it has finite number of extreme points. Let these be $x_1,\ x_2,\ ...,\ x_k \in S$. Clearly $S =< x_1,\ x_2,\ ...,\ x_k >$. Therefore, any $x \in S$ can be expressed as a convex combination of the extreme points, say

$$x = \sum_{j=1}^{k} \alpha_j x_j;\ \alpha_j \geq 0,\ \sum \alpha_j = 1 \quad \text{for all } j = 1,\ 2,\ ...,\ k$$

Let $\quad z_0 = \max.\ \{c^T x_j j = 1,\ 2,\ ...,\ k\}$. Then for any $x \in S$,

$$z = c^T x = c^T \left(\Sigma \alpha_j x_j\right) = \Sigma \alpha_j \left(c^T x_j\right) \leq \Sigma \alpha_j z_0 = z_0$$

$\therefore \quad z \leq z_0$ for any $x \in S$.

Thus, the maximum value of z is attained only at one of the extreme points of S. That is, at least one extreme point of S yields an optimal solution.

Now, since each extreme point of S corresponds to a basic feasible solution of the L.P.P. therefore, at least one basic feasible solution is optimal. This completes the proof.

Theorem-3 (Conditions of Optimality). A sufficient condition for a basic feasible solution to an L.P.P. to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $a_j \in A$ is not in the basic. B.

Proof. Let the L.P.P. be to determine x so as to

Maximize $z = cx;\ c,\ x^T \in R^n$ subject to the constraints :

$$Ax = b \text{ and } x \geq 0$$

where A and b are $m \times n$ and $m \times 1$ real matrices respectively. Let $\rho(A) = m$ so that we can choose an sub-matrix B of A as a basic matrix.

Let us assume that there exists a basic feasible solution x_B to this L.P.P. Let c_B be the cost vector corresponding to the basic variables.

Then $\quad Bx_B = b,\ x_B \geq 0$ and $z_0 = c_B x_B$

Now, for all those j for which $a_j \notin B,$ we are given that $z_j - c_j \geq 0$.

Let $a_j = b_j$ for all such j for which $a_j \in B.$ Then

$$y_j = B^{-1} b_j = e_j, \text{ the unit vector} \qquad (\text{since } y_j = B^{-1} a_j)$$

and

$$z_j - c_j = c_B y_j - c_j = c_B c_j - c_j$$

$$= c_{Bj} - c_j = 0 \qquad (\text{since } a_j \in B,\ c_{Bj} = 0)$$

Thus, $z_j - c_j \geq 0$ for all j for which $a_j \in A$.

Now, let x be a feasible solution. Then

$$\sum_{j=1}^{n}\left(z_j - c_j\right)x_j \geq 0,$$

or $$\sum_{j=1}^{n} z_j x_j \geq \sum_{j=1}^{n} c_j x_j \quad or \quad \sum_{j=1}^{n} c_B y_j x_j \geq \sum_{j=1}^{n} c_j x_j \qquad \text{(since } z_j = c_B y_j\text{)}$$

or $$\sum_{i=1}^{m} c_{Bi} \sum_{j=1}^{n} y_{ij} x_j \geq \sum_{j=1}^{n} c_j x_j, \qquad \text{(since } c_B y_j = \sum_{i=1}^{m} c_B y_j\text{)}$$

for all j for which $a_j \notin B$.

Now, since $$x_B = B^{-1}\left(Ax\right) = \left(B^{-1}A\right)x = Yx$$

or $$x_{Bi} = \sum_{j=1}^{n} y_{ij} x_j \quad i = 1, 2, ..., m$$

therefore, the above inequation can be written as

$$\sum_{i=1}^{m} c_{Bi} x_{Bi} \geq \sum_{j=1}^{n} c_j x_j \; or \; c_B x_B \geq cx \; or \; z_0 \geq z^*$$

where z^* is the value of the objective function for the feasible solution x.

Hence, z_0 is an optimum for that basic feasible solutions for which $z_j - c_j > 0$ for all j such that $a_j \notin B$.