M.S c Mathematics –SEM 2 Functional Analysis-L-2 CC-11 Unit 1

E-content 1–Dr Abhik Singh,

Guest faculty, PG Department of Mathematics, Patna University, Patna

Linear manifold spanned by Y : Let Y be any subset of a linear space S ,and let M be the set of all finite linear combination of vectors extracted from Y .Then M is called the linear manifold spanned by Y (or the linear hull of Y,or the linear envelope of Y) and is denoted by LM (Y) .

Theorem(1):

Let Y be any subset of a linear space S, and let M be the set of all finite linear combination of vectors extracted from Y. Then M is the smallest linear manifold containing Y.

Proof: (a) M is a linear manifold: If f and g are two vectors of M , which are therefore the linear combinations

$f = \sum_{i=1}^{n} c_i f_i$ and $g = \sum_{i=1}^{m} d_i g_i$

of vectors $f_1, f_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots f_n$ and

$g_1, g_2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots g_n$ of Y, then

$$af + bg = \sum_{i=1}^{n} a\, c_i f_i \quad + \sum_{i=1}^{m} \; d_i g_i$$

is also a linear combinations of vectors of Y ,and is hence in M .Therefore M is a Linear Manifold

(b) **M is the smallest linear manifold**:

If N is any other linear manifold containing Y ,then N contains all the finite linear combinations of vectors of Y,so that it contains M as a subset .Hence M is the smallest linear manifold containing Y.

Bases and Dimension:

Base:

A subset B of S is called a Hamel base (or simply base) for S if and only if B is a linearly independent set and LM(B) = S .

A linear space S is called finite –dimensional if and only if S has a finite base B , i. e B is a finite set which is a Hamel base .If S is not finite dimensional it is called infinite dimensional.

Dimension:

If S is a finite –dimensional space then its dimension is defined to be the number of elemens in any of its Hamel bases.

Example :

C^n has dimension n , ($e_1, e_2, e_3,$.........e_n) is a Hamel Base with n elements.

Theorem(2):

Every linear space S has a Hamel base.

Proof : Let P be the class of all linearly independent subset of S .Then P iss non-empty since the empty set lies in P.

Partially order P by set inclusion and let T={L_α} be a totally ordered subset of P.Let L = UL_α be the union of all the L_α in T.Then L is a linearly independent set.

For,if Y={S_1, S_2, S_3.......................S_n) is a finite subset of L then $S_i \in L_{\alpha_i}$,$1 \leq i \leq n$. By the total order in T we may arrange by relabeling if necessary ,that $L_{\alpha_1} \subset L_{\alpha_2} \subset L_{\alpha_3} \subset$...........$\subset L_{\alpha_n}$.

Hence $S_i \in L_{\alpha_i}$,$1 \leq i \leq n$,so that Y is linearly independent ,being a finite subset of the linearly independent set L_{α_n} . Thus T has L as an upper bound.

By Zorn,s lemma ,P has a maximal element ,B .Now every f in S is also in LM(B) .For ,if not ,then there exists an f in S but not in LM(B),whence B U{f} is linearly independent say . But B U {f} $\supset$B ,strictly , contrary to the fact that B is maximl.

From this we have shown that B is linearly independent and that LM(B) =S .Hence B is the required Hamel base.