**e-content**

By

Dr ABHAY KUMAR (Guest Faculty)

P.G.Department of Mathematics

Patna University,Patna

SEMESTER-2 ,CC-09 (Topology)

Topic: The Tietze Extension Theorem

**Theorem(The Tietze Extension Theorem):**

**Any bounded continuous real function on a closed subset $F$ of a normal space X can be extended continuously to the whole space X preserving the same bounds**.

Proof:Let $f$ be a bounded continuous function on the closed subset $F$ of X .Then there exists a real number $k > 0$ such that $|f(x)| \leq k$ for all $x \in F$

Now consider the subsets $F_1$ and $F_2$ of $F$ defined by

$$F_1 = \left\{x \in F : f(x) \leq \frac{-k}{3}\right\} = f^{-1}\left[-k, \frac{-k}{3}\right]$$

$$and\ F_2 = \left\{x \in F : f(x) \geq \frac{k}{3}\right\} = f^{-1}[\tfrac{k}{3}, k]$$

Then $F_1$ and $F_2$ are disjoint,non-empty and closed set in $F$

Since $f$ is continuous and since $F$ is closed so

$f^{-1}\left[-k, \frac{-k}{3}\right]$ and $f^{-1}\left[\frac{k}{3}, k\right]$

*i.e* $F_1$ and $F_2$ *are also closed in* $X$. *Since* $X$ *is normal*

$$so\ there\ exists\ a\ continuous\ function$$

$$g_1 : X \to \left[\frac{-k}{3}, \frac{k}{3}\right]\ such\ that$$

$$g_1(F_1) = \left\{\frac{-k}{3}\right\}\ and\ g_2(F_2\ ) = \left\{\frac{k}{3}\right\}$$

$$Now\ we\ define\ a\ function$$

$$h_1 on\ F\ by\ h_1(x) = f(x) - g_1(x)$$

Since $f\ and\ g_1\ are\ continuous\ so\ h_1\ is\ also\ continuous$

again $|h_1(x)| \leq \frac{2}{3}k\ if\ x \in F_{1,}then - k \leq f(x) \leq \frac{-k}{3}\ g_1(x) = \frac{-k}{3}$

$$Hence\ \frac{-2}{3}k \leq f(x) - g_1(x) \leq 0$$

i.e $\frac{-2}{3}k \leq h_1(x) \leq 0 \leq \frac{2}{3}k.$

$if\ x \in F_{2\ ,}then \frac{k}{3} \leq f(x) \leq k$ and$g_1(x) = \frac{k}{3}$

$$\frac{-2}{3}k \leq f(x) - g_1(x) \leq 0$$

$$\frac{k}{3} - \frac{k}{3} \leq f(x) - g_1(x) \leq k - \frac{k}{3}$$

$$0 \leq f(x) - g_1(x) \leq \frac{2k}{3}$$

i.e 0$\leq h_1(x) \leq \frac{2k}{3}\ \ finaly$ if $x \in F$

$$but\ x \notin F_2 \cup F_2$$

Then$\frac{-k}{3} < f(x) < \frac{k}{3}\ and \frac{-k}{3} \leq g_1(x) \leq \frac{k}{3}$

So that$\frac{-k}{3} - \frac{k}{3} < f(x) - g_1(x) < \frac{k}{3} - (\frac{-k}{3})$

$\frac{-2k}{3} < f(x) - g_1(x) = h_1(x) < \frac{2k}{3}$ So $|h_1(x)| \leq \frac{2k}{3}$

Applying the above procedur to $h_1(x)$ with bounds $\frac{-2k}{3}$ and $\frac{2k}{3}$ a continuous function $g_2(x)$ Is obtained on the whole space X with $|g_2(x)| \le \frac{1}{3} . \frac{2}{3} k$ *and a continuous function* $h_2(x) = h_1(x) - g_2(x)$ *is defined on* $F$ $|h_2(x)| \le (⅔)^2 k$. *In general*

, *we obtain for each positive integer* $n$, *a continuous*

*function* $g_n(x)$ *on* $X$ *with* $| g_n(x)| \le \frac{1}{3}(⅔)^{n-1}k$

*and a continuous function* $h_n(x) = h_{n-1}(x) - g_n(x)$ *on* $F$

*with* $|h_n(x)| \le (⅔)^n k$ *So by Weierstrass* $M -$ *test*

The series $\sum_{n=1}^{\infty} g_n(x)$ of continuous functions converges uniformly

On X and so defined a continuous function $f_0(x)$ *on* $X$

*with* $|f_0(x)| \le \frac{1}{3}\sum_0^{\infty}(⅔)^n k = k.$

Also on F we have $f_o(x) = g_1(x) + \sum_{n=1}^{\infty} g_{n+1}(x)$

$= f(x) - h_1(x) + \sum_{n=1}^{\infty}\{h_1(x) - h_{n+1}(x)\}$

$= \lim_{n\to\infty}\{ f(x) - h_{n+1}(x)\}$

Since $|h_{n+1}(x)| \le (⅔)^{n+1}\, k$ *and* $\lim_{n\to\infty}\{ h_{n+1}(x) = 0$

Hence $f_0(x) = f(x)$ *on* $F$. Thus there exists a continuous function $f_0(x)$ on X which is an extension of the given continuous function bounded function $f(x)$

on F and $f_0$ has the same bounds .It proves.