LPP (Simplex Method)
(M.Sc. Sem-III)

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Definition (Basic Solution) : Given a system of m simultaneous linear equations in n unknowns $(m < n)$

$$Ax = b,\ x^T \in R^n,$$

Where A is an $m \times n$ matrix of rank m. Let B be any $m \times n$ submatrix, formed by m linearly independent columns of A. Then, a soluton obtained by setting $n - m$ variables not associated with the columns of B, equal to zero, and solving the resulting system, is called a **basic solution** to the given system of equations.

The m variables, which may be all different from zero, are called **basic variables.** The $m \times m$ non-singular sub-matrix B is called a **basic matrix** with the columns of B as **basic vectors**.

Remarks : The name basic solution, as used above, merits a word of caution. If B is the basis sub-matrix chosen, then the basic solution to the system is

$$x_B = B^{-1}b.$$

But $x_B^T \in R^m$, and as such cannot be called a solution of the given system. Truly speaking, if x_B is a basic solution, then a solution to the given system is $\left[x_B^T,\ 0\right]$ where $x_B^T \in R^m$, and $0 \in R^{n-m}$. However, we shall follow the current usage and call x_B a basic solution of the given system, remembering all the time that the actual solution is $\left[x_B^T,\ 0\right]$.

Definition (Degenerate solution). A basic solution to the system is called **degenerate** if one or more of the basic variables vanish.

SAMPLE PROBLEMS

1. Obtain all the basic solutions to the following system of linear equation :

$$x_1 + 2x_2 + x_3 = 4$$

$$2x_1 + x_2 + 5x_3 = 5.$$

Sol. The given system of equations can be written in the matrix form as $Ax = b,$ where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Since, rank of A is 2, the maximum number of linearly independent columns of A is 2. Thus, we can take any of the following, 2×2 sub-matrices as basic matrix B :

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

The variables not associated with the columns of B are $x_3,\ x_2$ and x_1 respectively, in the three different cases.

Let us first take $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A basic solution to the given system is now obtained by setting $x_3 = 0,$ and solving the system.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Thus, a basic (non-basic) solution to the given system is :

$$(\text{Basic})\, x_1 = 2,\ x_2 = 1; \qquad (\text{Non-basic})\, x_3 = 0$$

Similarly, the other two basic and non-basic solutions are :

$$(\text{Basic})\, x_1 = 5,\ x_2 = -1; \qquad (\text{Non-basic})\, x_2 = 0$$

and $(\text{Basic})\, x_2 = 5/3,\ x_3 = 2/3; \qquad (\text{Non-basic})\, x_1 = 0$

We observe that all the above three basic solutions are non-degenerate.

2. Show that the following system of linear equations has a degenerate solution :

$$2x_1 + x_2 - x_3 = 2$$

$$3x_1 + 2x_2 + x_3 = 3.$$

Sol. The given system of equations can be written as $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Since rank of A is 2, the maximum number of linearly independent columns of A is 2. Thus, we can take any of the following 2×2 sub-matrices of A, as basic matrix B :

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix},\ \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$$

The variables not associated with the columns of these sub-matrices are, respectively, x_3, x_1 and x_2.

Considering $B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, a basic solution to the given system is obtained by setting $x_3 = 0$ and solving the system

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus, basic solution to the given problem is

$$(\text{Basic})\, x_1 = 1,\ x_2 = 0; \qquad (\text{Non-basic})\, x_3 = 0$$

Similarly, the other two solutions are :

$$(\text{Basic})\, x_2 = 5/3,\ x_3 = -1/3; \qquad (\text{Non-basic})\, x_1 = 0.$$

and $(\text{Basic})\, x_1 = 1,\ x_3 = 0 \qquad (\text{Non-basic})\, x_2 = 0$

in each of the two basic solutions, at least one of the basic variables is zero. Hence, two of the basic solutions are degenerate solutions.

Definition (Basic feasible solution). A feasible solution to an L.P.P., which is also a basis solution to the problem is called a **basic feasible solution** to the L.P.P.

Illustrations

1. In sample problem 1 observe that $[5,\ 0,\ -1]$ is not a feasible solution. Only basic feasible solutions are :

(i) $x_2 = 5/3$ and $x_3 = 2/3$; (ii) $x_1 = 2$ and $x_2 = 1$

2. In sample problem 2, the non-degenerate solution $[0, 5/3, -1/3]$ is not feasible. Only basic feasible degenerate solutions are :

(i) $x_1 = 1$ and $x_2 = 0$ (ii) $x_1 = 1$ and $x_3 = 0$.

Definition (Associated cost vector). Let x_B be a basic feasible solution to the L.P.P. :

$$\text{Maximize } z = cx \text{ subject to : } Ax = b \text{ and } x \geq 0$$

Then, the vector $$c_B = (c_{B1}, c_{B2}, ..., c_{Bm})$$

where c_{B1} are components of c associated with the basic variables, is called the **cost vector associated** with the basic feasible solution x_B.

It is obvious that the value of the objective function for the basic feasible solution x_B, is given by

$$z_0 = c_B x_B .$$

Definition (Improved basic feasible solution). Let x_B and $\hat{x}_B$ be two feasible solutions to the standard L.P.P. Then $\hat{x}_B$ is said to be an **improved basic feasible solution,** as compared to x_B, if

$$\hat{c}_B \hat{x}_B \geq \hat{c}_B \hat{x}_B$$

where $\hat{c}_B$ is constituted of cost components corresponding to $\hat{x}_B$,

Definition (Optimum basic feasible solution). A basic feasible solution x_B to the L.P.P. :

$$\text{Maximize } z = cx \text{ subject to : } Ax = b \text{ and } x \geq 0$$

is called an **optimum basic feasible solution** if $z_0 = c_B x_B \geq z^*$ where z^* is the value of objective function for any feasible solution.

PROBLEMS

3. Find all the basic feasible solutions of the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$