M.S c Mathematics –SEM 2 Functional Analysis CC-11 Unit 1

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Linear Space

Let S be a set : It will be called a space and its elements will be called point or vectors and will usually be denoted by f,g,h, etc

Linear space : S is termed a linear space (or vector space) if

(a)An operation called addition and denoted by + is defined in S ,which assigns to every two vectors f,g of S ,a vector f+ g of S ,called the sum of f and g, satisfying the following properties : for any vector f,g , h ,of S .

(i)f + g = g + f (commutative)

(ii) (f + g) + h= f + (g + h)(associative)

(iii) the equation g + x = f hs at least one solution x in S

(b) A certain scalar system either the complex number system or the real number system (the elements of the scalar system

being called scalars) , an operation called multiplication by a scalar is defined, i. e to every scalar a and every vector f, a vector af called the scalar multiple of f by a ,satisfying the following properties : for any scalars a, b, c and any vectors f, g, h

(i)a(bf)= (a b)f (associative with respect to scalar, multiplication to scalar)

(ii) a (f+ g) = a f + a g (distributivity of scalar multiplication over addition of vectors)

(iii)1f=f (property of multiplication by unity)

The linear space is called a complex linear space or a real linear space according as the prescribed scalar system is the complex number system or the real number system.

1. Linear dependence : n vectors $f_1, f_2, \ldots\ldots\ldots f_n$,are said to be linearly dependent if some one of them say f_k ,can be expressed in terms of the remaining ones in the form

$$f_k = a_1 f_1 + a_2 f_2 + \cdots\cdots\cdots + a_n f_n$$

2. **Linear Combination** : For Vectors $f_1, f_2, \ldots\ldots\ldots\ldots f_n$, any vector of the form

$c_1 f_1 + c_2 f_2 + \ldots\ldots\ldots\ldots + c_n f_n$ is termed a linear combination of $f_1, f_2, \ldots\ldots\ldots\ldots f_n$ (i.e a combination obtained by applying addition and multiplication by scalars which are the operations prescribed in the definition of linear spaces.

3. **Linear Independence** : The negative counterpart of linear dependence is called linear independence of vectors $f_1, f_2, \ldots\ldots\ldots\ldots f_n$, which is said to hold if none of these vectors can be expressed as a linear combination of the others,the relation

$$c_1 f_1 + c_2 f_2 + \ldots\ldots\ldots\ldots + c_n f_n = 0$$

can hold only for the case

$$c_1 = c_2 = \ldots\ldots\ldots\ldots\ldots\ldots = c_n$$

This means that if vectors $f_1, f_2, \ldots\ldots\ldots\ldots f_n$ are linearly independent and if scalars vectors $b_1, b_2, \ldots\ldots\ldots\ldots b_n$ are chosen such that all of b_i's are not zero, then

$$b_1 f_1 + b_2 f_2 + \cdots\cdots\cdots + b_n f_n \neq 0$$

Linear Manifold

Let M be a subset of the linear space S. It is of interest to know whether M itself is a linear space with the same operations as in S. Thus in case of three-dimensional vector space ,every plane or straight line passing through the origin is itself a vector space ,but the same cannot be said of arbitrary sets, such as spheres or cubes in the space, as the sum of two vectors taken from the set may fall outside the set.

Linear manifold (or subspace) :

A subset M of a linear space is called linear manifold(or a subspace) if

(i) it contains all the sums of all the vectors contained in it i.e if f and g are in M then so is f+g, and

(ii) it contains all the scalar multiples of all its vectors, i.e if f is in M then so is af for any scalar a.

Both (i) and (ii) can be combined into a single conditions by saying that if f and g are in M ,then so is a f + b g ,for any scalars a, b