Fredhlom integral equation

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Solution of Fredhlom integral equation by iterative method.

Theorem 1. *Consider Fredhlom integral equation of the second kind.*

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt \tag{1}$$

Proof. As azero -order approximation to the required solution $y(x)$,let us take

$$y_0(x) = f(x) \tag{2}$$

further,if $y_n(x)$andy_{n-1} are the nth order and $(n-1)th$ order approximations respectively,then these are connected by

$$y_n(x) = f(x) + \lambda \int_a^b K(x,t)y_{n-1}(t)dt \tag{3}$$

We know that the iterated kernels (or iterated function) $k_n(x,t), (n = 1, 2, 3, \dots$ are defined by

$$K_1(x,t) = K(x,t), \tag{4a}$$

$$k_n(x,t) = \int_a^b K(x,z)k_{n-1}(z,t)dz \tag{4b}$$

and

putting $n = 1$ in equation (3), the first-order approximation y_1 is given by

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t)y_0(t)dt \tag{5}$$

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But from equation (2

$$y_0(t) = f(t) \tag{6}$$

From equations (5) and (6), we get

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt \tag{7}$$

putting $n = 2$ in equation (3), the second-order approximation y_2 is given by

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) y_1(t) dt, \tag{8a}$$

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) y_1(z) dz \tag{8b}$$

Replacing x by z in equation (5), we get

$$y_1(z) = f(z) + \lambda \int_a^b K(z,t) f(t) dt \tag{9}$$

From equations (8a) and (9), we have

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) \Big[f(z) + \lambda \int_a^b K(z,t) f(t) dt \Big] dt,$$

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) f(z) dz + \lambda^2 \int_a^b K(x,z) \Big[\int_a^b K(z,t) f(t) dt \Big] dz \tag{10a}$$

On changing the order of the integration in third term on R.H.S. of equation (10a)

$$y_2(x) = f(x) + \lambda \int_a^b K(x,z) f(z) dz + \lambda^2 \int_a^b f(t) \Big[\int_a^b K(x,z) K(z,t) dz \Big] dt \tag{11}$$

or

$$y_2(x) = f(x) + \lambda \int_a^b K_1(x,t) f(t) dt + \lambda^2 \int_a^b K_2(x,t) f(t) dt \tag{12}$$

$$y_2(x) = f(x) + \sum_{m=1}^{m=2} \lambda^m \int_a^b K_m(x,t) f(t) dt \tag{13}$$

proceeding likewise, we easily obtain by Mathematical induction the n^{th} order approximation solution $y_n(x)$ of equation (1) as

$$y_n(x) = f(x) + \sum_{m=1}^{m=n} \lambda^m \int_a^b K_m(x,t) f(t) dt \tag{14}$$

Taking limitt both side as $n \to \infty$

$$y(x) = \lim_{n\to\infty} y_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t)f(t)dt \tag{15}$$

We now determine the resolvent kernal(reciprocal kernal) $R(x,t;\lambda)$ in the terms of the iterated kernals $k_m(x,t)$. For this equation (14) is uniformly convergent.

$$y(x) = \lim_{n\to\infty} y_n(x) = f(x) + \int_a^b \Big[\sum_{m=1}^{\infty} \lambda^m K_m(x,t)\Big] f(t)dt \tag{16}$$

comparing equation (17) with

$$y(x) = f(x) + \lambda \int_a^b R(x,t:\lambda)f(t)dt \tag{17}$$

Here

$$R(x,t:\lambda) = \sum_{m=1}^{\infty} \lambda^m K_m(x,t)$$

Determination of the conditions of convergence

Consider the partial sum equation (14) and apply the Schwarz inequality

$$|\int_a^b K_m(x,t)f(t)dt|^2 \leq \left(\int_a^b |K_m(x,t)f(t)dt|^2\right) \int_a^b |f(t)|^2 dt. \tag{18}$$

Let

$$D = \text{norm of } \; f(t) = \left[\int_a^b |f(t)|^2 dt\right]^{\frac{1}{2}} \tag{19}$$

Further, let C_m^2 denotes the upper bound of the integral

$$\int_a^b |K_m(x,t)f(t)dt|^2,$$

so that,

$$\int_a^b |K_m(x,t)f(t)dt|^2 \leq C_m^2 \tag{20}$$

Using equations (20) and (19), (18) reduces to

$$\left|\int_a^b K_m(x,t)f(t)dt\right|^2 \leq C_m^2 D^2 \tag{21}$$

now, applying Swartz inequality to relation

$$K_m(x,t)=\int_a^b K_{m-1}(x,z)K(z,t)dz,$$

We get

$$|K_m(x,t)|^2 \le \left(\int_a^b K_{m-1}(x,z)|^2 dz\right) \times \left(\int_a^b K(z,t)|^2 dz\right)$$

which when integrated with respect to t, gives

$$\int_a^b |K_m(x,t)|^2 \le B^2 C_{m-1}^2 \tag{22}$$

Where

$$B^2=\int_a^b\int_a^b |K_m(x,t)|^2 dxdt. \tag{23}$$

The inequality (22) gives rise to the recurrence relation

$$C_m^2 \le B^{2m-2} C_1^2 \tag{24}$$

Using equations (21) and (24), we get

$$\left|\int_a^b K_m(x,t)f(t)dt\right|^2 \le C_m^2 D^2 B^{2m-2} \tag{25}$$

showing that the general term of the partial sum (14) has a magnitude less than the quantity $DC_1|\lambda|^m B^{m-1}$. Hence the infinite series (15) converges faster than the geometric series with command ratio $|\lambda|B$. It follows that, if the condition

$$\Big|\lambda\Big| \le \frac{1}{\sqrt{\int_a^b\int_a^b \Big|K(x,t)\Big|^2 dxdt}} \tag{26}$$

is satisfied. Then the series (16) is uniformly convegent. **Uniqueness of solution for a given λ**

If possible, let equation(1) possess two solutions $y_1(x)$ and $y_2(x)$. Then we have

$$y_1(x) = f(x) + \lambda \int_a^b K(x,t) y_1(t) dt \tag{27}$$

$$y_2(x) = f(x) + \lambda \int_a^b K(x,t) y_2(t) dt \tag{28}$$

$$y_1(x) - y_2(x) = \phi(x) \tag{29}$$

Substracting equation (28) from (27), we have

$$y_1(x) - y_2(x) = \int_a^b K(x,t)(y_1(t) - y_2(t)) dt = \phi(x) \tag{30}$$

which is homogeneous integral equation. Applying the Schwarz inequality to equation (30), we have

$$|\phi(x)|^2 \leq |\lambda|^2 \left(\int_a^b |K(x,t)|^2 dt \right) \times \left(\int_a^b |\phi(t)|^2 dt \right) \tag{31}$$

integrating with respect to x

$$\int_a^b \left|\phi(x)\right|^2 dx \leq \left|\lambda\right|^2 \left(\int_a^b \int_a^b \left|K(x,t)\right|^2 dx \ dt \right) \times \left(\int_a^b \left|\phi(x)\right|^2 dx \right),$$

$$\text{or,} \quad \int_a^b \left|\phi(x)\right|^2 dx \leq \lambda^2 B^2 \int_a^b \left|\phi(x)\right|^2 dx,$$

$$\implies \quad \left(1 - \left|\lambda\right|^2 B^2\right) \int_a^b \left|\phi(x)\right|^2 dx \leq 0,$$

$$\therefore \quad \phi(x) = 0,$$

$$y_1(x) = y_2(x)$$

□