Complex Integration (M.Sc.)

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Evaluation of real definite integration by Cantour Integration

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of residues of } f(z) \text{ at the pole within } C\right)$$

use $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{1}{2i}\left[z-\dfrac{1}{z}\right]$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left[z+\frac{1}{z}\right]$$

where $z = re^{i\theta}$ and $|z| = r$

if $z = e^{i\theta} \Rightarrow dz = e^{i\theta} i\,d\theta$

$\Rightarrow \boxed{d\theta = \dfrac{dz}{iz}}$

Evaluate :

(i) $\displaystyle\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

Solution : Let $I = \displaystyle\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

$$= \text{Real part}\left[\frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+4\cos\theta}\,d\theta\right]$$

$$= \text{Real part}\left[\frac{1}{2}\int_0^{2\pi} \frac{1+2e^{i\theta}}{5+2\left(e^{i\theta}+e^{-i\theta}\right)}\right]d\theta$$

$= \text{putting } e^{i\theta} = z \qquad \text{get } d\theta = \dfrac{dz}{iz}$

$$I = \text{Real part}\left[\frac{1}{2}\int_C \frac{1+2z}{5+2\left(z+z^{-1}\right)}\frac{dz}{iz}\right]$$

$$= \text{Real part of}\left[\frac{1}{2}\int \frac{-i(2z+1)}{(2z+1)(z+2)}\,dz\right]$$

$$= \text{Real part of}\left[-\frac{1}{2}\int \frac{1}{z+2}\,dz\right]$$

Poles are : $z+2=0 \qquad z=-2$

$\Rightarrow$ No. of poles of $f(z)$ lies inside the circle C.

$\Rightarrow \displaystyle\int f(z)\,dz = 0$

$\Rightarrow \; I = \text{Real part } [0 + i_0] \qquad \boxed{I = 0}$

(2) Do yourself (using Contour Integration)

(a) Evaluate : $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$

Ans. $\frac{\pi}{6}$

(b) Evaluate : $\int_0^{\infty} \frac{\cos 3\theta}{5 + 4\cos\theta} d\theta$

** **Evaluation of** $\int\limits_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$

Where $f(x)$ & $g(x)$ are polynomials in *x*.

The given integral can be reduced to contour integrals. if

(i) $g(x)$ has no real root.

(ii) The degree of $f(x) >$ degree of $g(x)$ by at least two.

i.e. $\deg(f(x)) - \deg(g(x)) \geq 2$

Steps involved :

Let $h(x) = \frac{f(x)}{g(x)}$

Consider $\int_C h(z)\, dz$

Where C is a curve; consisting of upper half C_R of the circle $|z| = R$ and part of real axis from –R to +R.

If there are no poles of $f(z)$ on the real line, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its.

Circumference C_R in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's residue theorem

We have $\int_C f(z)\, dz = 2\pi i \sum \text{Residues}$

i.e., $\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = 2\pi i (\text{Sum of residue in C})$

$$\Rightarrow \int_{-R}^{R} f(x)\, dx = -\int_{C_R} f(z)\, dz + 2\pi i (\text{Sum of residues within } C)$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = -\lim_{R\to\infty} \int f(z)\, dz + 2\pi i \sum R$$

Now $\lim\limits_{R\to\infty} \int_{C_R} f(z)\, dz = \int_0^{\pi} f(\mathrm{Re}^{i\theta}) Rie^{i\theta} d\theta = 0$

Hence $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i (\text{Sum of residues in C})$