Osgood and Nagumo Theorem

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Theorem 1 (Osgood Uniqueness theorem) *Let $f(x,y)$ be a continuous function on $\overline{S} = \Big\{(x,y) \in \mathbb{R}^2 : \Big|x - x_0\Big| \le a,\ \text{and}\ \Big|y - y_0\Big| \le b\Big\}$. If $\forall\ (x, y_1),\ (x, y_2) \in \overline{S}$ such that*

$$\Big|f(x,y_1) - f(x,y_2)\Big| \le w\Big(\Big|y_1(x) - y_2(x)\Big|\Big) \tag{1}$$

*where $w(z)$ is the same as the Lemma (**??**). The IVP $y'(x),\quad y(x_0) = y_0$ has at most one solution in $\overline{S}$*

First proof lemma (**??**) then arises two condition according as $x \in [x_0 - a,\ x_0 + a]$

Case (i) Let $y_1(x)$ and $y_2(x)$ be two solution of the given IVP

$$y'(x), \quad y(x_0) = y_0, \quad \text{in}\ \ x_0 \le x \le x_0 + a \tag{2}$$

We have

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt$$

and

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))dt$$

$$\begin{aligned}
\therefore \Big|y_1(x) - y_2(x)\Big| &= \Big|\int_{x_0}^{x} f(t, y_1(t))dt - \int_{x_0}^{x} f(t, y_2(t))dt\Big| \\
&\le \int_{x_0}^{x} \Big|f(t, y_1(t)) - f(t, y_2(t))\Big|dt \\
&\le \int_{x_0}^{x} w\Big(\Big|y_1(t) - y_2(t)\Big|\Big)dt
\end{aligned}$$

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Now, let $u(x) = \Big|y_1(x) - y_2(x)\Big|$ then $u(x)$ satisfies the condition in the lemma (**??**).

$$u(x) = 0 \quad \forall \quad x \in \Big[x_0, x_0 + a\Big]$$

Case (ii) Again to prove the theorem (1) $\forall \;\; x \in \Big[x_0 - a, x_0\Big]$.

we need to prove that

$$u(x) = \Big|y_1(x_0 - x) - y_2(x_0 - x)\Big| \tag{3}$$

Let $u(x)$ be s non-negative continuous function in $\Big|x - x_0\Big| \leq a$ such that $\quad u(x_0) = 0 \quad u\,'(x_0) = 0$ ($u(x)$ is differentiable), then the inequality

$$u(x) \leq \Big|\int_{x_0}^{x} \frac{u(t)}{t - x_0} dt\Big|$$

$$\implies \quad u(x) = 0 \quad \forall \;\; x \in \Big[x_0 - a, x_0 \;\; + a\Big]$$

Let

$$v(x) = \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt \tag{4}$$

then from statement

$$u(x) \leq v(x)$$

and the integral given by equation (4) exists.

Differentiating w.r.t. x of equation (4) under Leibnitz's rule,

hence

$$v'(x) = \frac{u(x)}{x - x_0}$$

$$\implies \quad v'(x) = \frac{u(x)}{x - x_0} \leq \frac{v(x)}{x - x_0}$$

$$\implies \quad v'(x) \leq \frac{v(x)}{x - x_0}$$

$$\implies \quad \frac{v'(x)}{x - x_0} - \frac{v(x)}{(x - x_0)^2} \leq 0$$

$$\implies \quad \frac{d}{dx}\Big(\frac{v'(x)}{x - x_0}\Big) \leq 0$$

$$\text{Since} \quad x \geq x_0 \quad \implies \quad v(x) \leq v(x_0)$$

$\implies$ $v(x)$ is monotonic decreasing at x_0

$$v(x) \leq 0 \tag{5}$$

Now,

$$\lim_{x\to x_0}\frac{u(x)}{x-x_0}=u'(x_0)$$

$$\text{Since}\ \ u(x)\ge 0$$

$$v'(x)=\frac{u(x)}{x-x_0}\ge 0$$

$$v(x)\ge 0 \tag{6}$$

From equations (5) and (6)

$$\therefore\quad v(x)=0 \tag{7}$$

From equations (4) and (7)

$$\int_{x_0}^{x}\frac{u(t)}{t-x_0}dt \qquad \Longrightarrow \qquad u(x)=0\quad\forall\ \ x\in\Big[x_0-a,x-0+a\Big]$$

Hence **proved**

Theorem 2 (Nagumo uniqueness theorem) *Let $f(x,y)$ be a continuous function on $\overline{S}=\Big\{(x,y)\in\mathbb{R}^2:\Big|x-x_0\Big|\le a,$ and $\Big|y-y_0\Big|\le b$ and $\forall\ \ (x,y_1),\ \ (x,y_2)\in\overline{S}\Big\}$ such that*

$$\Big|f(x,y_1)-f(x,y_2)\Big|\le K\Big|x-x_0\Big|^{-1}\Big|y_1(x)-y_2(x)\Big| \tag{8}$$

where $K\le 1,\ \ x\ne x_0$. Then the IVP $\ y'(x)=f(x,y),\ \ \ y(x_0)=y_0\ $ has at most one solution in $\overline{S}$

Let $y_1(x)$ and $y_2(x)$ be two solution of the given IVP

$$\text{Since}\ \ \Big|y_1(x)-y_2(x)\Big|=\Big|\int_{x_0}^{x}f(t,y_1(t))dt-\int_{x_0}^{x}f(t,y_2(t))dt\Big|$$

$$\le\int_{x_0}^{x}\Big|f(t,y_1(t))-f(t,y_2(t))\Big|dt$$

$$\le K\Big|\int_{x_0}^{x}\frac{\Big|y_1(t)-y_2(t)\Big|}{t-x_0}\Big|dt$$

$$\Longrightarrow\qquad \Big|y_1(x)-y_2(x)\Big|\le K\Big|\int_{x_0}^{x}\frac{y_1(t)-y_2(t)}{t-x_0}\Big|dt$$

$$\therefore\quad \Big|y_1(x)-y_2(x)\Big|\le\Big|\int_{x_0}^{x}\frac{y_1(t)-y_2(t)}{t-x_0}\Big|dt$$

If we have $u(x)=\Big|y_1(x)-y_2(x)\Big|$ then we have

$$u(x_0) = 0 \quad \textbf{and} \quad u'(x_0) = 0$$

$$u(x) \leq \Big| \int_{x_0}^{x} \frac{u(t)}{t - x_0} \Big|$$

is also satisfied

$$u(x) = 0 \quad \forall \quad \Big| x - x_0 \Big|$$

we set $u(x) = \Big| y_1(x) - y_2(x) \Big|$

$\implies$ $u(x)$ is non-negative and continuous $\forall$ $\Big| x - x_0 \Big|$ and $u(x_0) = 0$ i.e., $y_1(x_0) = y_2(x_0)$

$$\begin{aligned}
u'(x) &= \lim_{h \to 0} \frac{u(x_0 + h) - u(x_0)}{h} \\
&= \lim_{h \to 0} \frac{\Big| y_1(x_0 + h) - y_2(x_0 + h) \Big| - \Big| y_1(x_0) - y_2(x_0) \Big|}{h} \\
&= \lim_{h \to 0} \frac{\Big| y_1(x_0) + h y_1'(x_0 + \theta h) - y_2(x_0) - h y_2'(x_0 + \theta h) \Big|}{h}
\end{aligned}$$

(by Taylor's expansion)

$$\begin{aligned}
&= \lim_{h \to 0} \frac{\Big| h \Big| \Big| y_1'(x_0 + \theta h) - y_2'(x_0 + \theta h) \Big|}{h} \\
&= \lim_{h \to 0} \frac{\Big| h \Big|}{h} \lim_{h \to 0} \Big| y_1'(x_0 + \theta h) - y_2'(x_0 + \theta h) \Big|
\end{aligned}$$

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$$u'(x) = 0$$

so all the three conditions of lemma are satisfied, then $u(x) = 0 \quad \forall x$ in $\Big| x - x_0 \Big| \leq a$

$$\implies \quad \Big| y_1(x) - y_2(x) \Big| = 0 \quad \forall \quad \Big| x - x_0 \Big| \leq a$$

Hence **proved**

...............All the best.........