Measure Theory (M.Sc. Sem-II)

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1. The cantor ternary set :

Let us consider the interval $I = [0, 1]$

Now remove

$$I_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$$

from I when we get as residual

residual are : $J_{1,1} = \left[0, \frac{1}{3}\right]$ & $J_{1,2} = \left[\frac{2}{3}, 1\right]$

again remove

$$I_{2,1} = \left(\frac{1}{9}, \frac{2}{9}\right) \text{ \& } I_{2,2} = \left(\frac{7}{9}, \frac{8}{9}\right)$$

from residuals $J_{1,1}$ and $J_{1,2}$

then we have $\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$

As residuals

On continuining this process we arrive at the residual closed intervals at n[th] stage.

$J_{n,1}$ .... $J_{n,2^n}$ each of length $\frac{1}{3^n}$ the open interval (removed).

$I_{n,r}$ also being of length $\frac{1}{3^n}$, if we write $P_n = \bigcup_{k=1}^{2^n} J_{n,k}$ then,

$P = \bigcap_{n \geq 1} P_n$ is defined as Cantor ternary set.

Note :– The cantor ternary set is uncountable.

How

If possible we suppose the set P (Cantor ternary set) is countable and

Let $x^{(1)}, x^{(2)}, ...$ be the enumeration of P

But are known the element in P is as of the form $x = 0 \cdot x_1 x_2 ....$ with

$x_n \in \{0, 2\}$

If $x_n^{(n)} = 0$ let $x_n = 2$; if $x_n^{(n)} = 2$

Let $x_n = 0$, then $x = 0 \cdot x_1 x_2 ....$ differ from

each $x^{(n)}$ but $x \in P$ so no enumeration is possible.

$\Rightarrow$ P is uncountable.

(2) The Lebegue function

Let $J_{n,k}$ and $I_{n,k}$ are defined as in cantor set P.

and L_n is monotone increasing on $[0, 1]$ $\forall$ $n \in N$

Which is linear and increased by $\frac{1}{2^n}$ with

$$L_n(0) = 0, \qquad L_n(1) = 1 \qquad \& \quad L_n(x) = \text{const.}\ \forall\, x \in I_{n,k}$$

$\Rightarrow \quad n > m$

we have $\left|L_{n(x)} - L_m(x)\right| < \frac{1}{2^m}$ $\qquad \Rightarrow \{L_n$ is cauchy's sequence$\}$

then the function $L(n) = \lim\limits_{n\to\infty} L_n(x)$

is defined as lebegue function

MEASURE ON REAL LINE

Alert : All the sets over which we are going to define a measure are subset of real line $\mathbb{R}$

We will be concerned particularly with interval I of the form $I = [a, b)$ where a & b are finite.

If $a = b \quad \Rightarrow I = \phi$ (Empty set)

Definition : The lebegue outer measure of a set A is given by

$$m^*(A) = \inf \sum \ell(I_n) \text{ where}$$

Infimum is taken over all finite or countable collections of intervals $\{I_n\}$ such that $A \subseteq \bigcup\limits_{n\geq 1} I_n$

or another may we can say

$\bigcup I_n$ is countable covering of A

Properties :-

(i) $m^*(A) \geq 0 \qquad \forall\ A \subseteq \mathbb{R}$

(ii) $m^*(\phi) = 0$

(iii) $m^*(A) \leq m^*(B) \forall\ A \subseteq B$

(iv) $m\{x\} = 0 \qquad$ for all $x \in R$

Proof : (i) By the definition (i), (ii) and (iii) are obvious.

for (iv) Let $x \in I_n$

where $I_n = \left[x,\ x + \frac{1}{n}\right) \forall n$

and $\ell(I_n) = \frac{1}{n}$

$$m^*\{x\} = \inf \sum_{n\geq 1} \frac{1}{n} \quad = 0 \qquad\qquad \text{as } n \to \infty$$

$$\boxed{m^*\{x\} = 0}$$