MEASURE THEORY : (M.Sc. Sem-II)
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TOPIC-4

THE FOUR DERIVATES

Definition :– (i) If f is an extended real valued finite at x and defined in a open interval containing x, then the following four quantities : not necessarily finite are called respectively the upper right derivate the lower derivate, the upper left derivate and lower left derivate

$$D^{+}f(x)=\lim_{h\to 0}\sup\frac{f(x+h)-f(x)}{h};\ \text{upper right derivate}$$

$$D_{+}f(x)=\lim_{h\to 0}\inf\frac{f(x+h)-f(x)}{h};\ \text{lower right derivate}$$

$$D^{-}f(x)=\lim_{h\to 0}\sup\frac{f(x-h)-f(x)}{-h};\ \text{upper left derivate}$$

$$D_{-}f(x)=\lim_{h\to 0}\inf\frac{f(x-h)-f(x)}{-h}:\ \text{lower left derivate}$$

Note : $D^{+}f(x)\geq D_{+}f(x)$ & $D^{-}f(x)\geq D_{-}f(x)$

the function f is differentiable at x if and only if the four derivates have a finite common value which we then write as usual $f'(x)$

Example (1) : Let $f(x)=|x|$ find D^{+}, D_{+}, D^{-} & D_{-} and prove that f is not differentiable.

Solution :–

$$D^{+}f(0)=\lim_{h\to 0}\sup\frac{f(h)-f(0)}{h}=\frac{|h|}{h}=1$$

$$D_{+}f(0)=\lim_{h\to 0}\inf\frac{f(h)-f(0)}{h}=\frac{|h|-0}{h}=1$$

$$D^{-}f(0)=\lim_{h\to 0}\sup\frac{f(h)-f(0)}{-h}=\frac{-|h|}{h}=-1$$

$$D_{-}f(0)=-1$$

Thus we have $D^{+}=D_{+}=1$ & $D^{-}=D_{-}=-1$

$\Rightarrow$ $f'(0)$ does not exist.

Example (2) : Evaluate four derivatives of $f(x)$ at $x=0$ where $f(x)$ is given by

$$f(x)=\begin{cases} a\sin^2\frac{1}{x}+bx\cos^2\frac{1}{x} & ; \quad x>0 \\ 0 & ; \quad x=0 \\ a'x\sin^2\frac{1}{x}+b'x\cos^2\frac{1}{x} & ; \quad x<0 \end{cases}$$

where $a<b,\ a'<b'$

Solution :– We have

$$D^+f(0)=\lim_{h\to 0}\sup\left[a\sin^2\frac{1}{h}b\cos^2\frac{1}{h}\right]=b$$

$$D_+^-f(0)=\lim_{h\to 0}\inf\left[a\sin^2\frac{1}{h}+b\cos^2\frac{1}{h}\right]=a$$

$$D^-f(0)=\lim_{h\to 0}\sup\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\sup\frac{\left[a'(-h)\sin^2\frac{1}{h}-b'h\cos^2\frac{1}{h}\right]}{-h}$$

$$=b'$$

$$D_-f(0)=\lim_{h\to 0}\inf\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{\left[a^1(-h)\sin^2\frac{1}{h}-b'h\cos'^2\frac{1}{h}\right]}{-h}$$

$$=a'$$

Assignment : Evaluate the four derivates of $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right); & x\neq 0 \\ 0 \quad ; & x=0 \end{cases}$

at $x=0$

Assignment (2) show that if $f'(x)$ exists then $D^+(f+g)(x)=f'(x)+D^+g(x)$

Assignment (3) Give an example where $D^+(f+g)\neq D^+f+D^+g$

FUNCTION OF BOUNDED VARIATION

Let f is defined & finite valued on finite interval $[a,\ b]$ let $P[a,\ b]=\{a=x_0<x_1,<x_2,\ .....x_n=b\}$ be a partition of $[a,\ b]$

Put $p=\sum_{i=1}^{k}\left(f(x_1)-f(x_{i-1})\right)^+$, $n=\sum_{i=1}^{k}\left(f(x_i)-f(x_{i-1})\right)^-$

& $t=p+n=\sum_{i=1}^{k}\left|f(x_i)-f(x_{i-1})\right|$,

where $\alpha^{+} = \max(\alpha, 0)$, $\alpha^{-} = \max(-\alpha, 0)$

so $t, p, n \geq 0$ and $f(b) - f(a) = p - n$

Also put

$$T_f[a, b] = \sup t = \sup_{\forall p} \sum_{i=1}^{k} \left| f(x_i) - f(x_{i-1}) \right|$$

where supremum is taken over all partitions of $[a, b]$

Now function f over $[a, b]$ is said to be function of bbd variation iff $T_f[a, b] < \infty$

Also, we will use $\boldsymbol{p} = \sup p$, $\boldsymbol{N} = \sup n$, $\boldsymbol{T} = \sup t$

defined as positive, negative and total variations, of f on $[a, b]$

A function is said to belongs to $BV[a, b]$ if $[a, b] < \infty$

Theorem (1) Let $f \in BV[a, b]$ then $f(b) - f(a) = P - N$ and $T = P + N$ all variations being on the finite interval $[a, b]$

Proof : for any partition $f(b) - f(a) = p - n$

$\Rightarrow p = n + f(b) - f(a) \leq N + f(b) - f(a)$

on taking supremum

$p \leq N + f(b) - f(a)$

similarly : $n = p + f(a) - f(b)$

gives $N \leq f(a) - f(b) + P$

But $P - N \leq f(b) - f(a) \leq P - N$

$\Rightarrow \boxed{f(b) - f(a) = P - N}$ proved.

Also $T \geq p + n = 2p - f(b) + f(a) = 2p + N - P$

on taking supremum

$\Rightarrow T \geq P + N$ but $t = n + p \leq N + P$

similarly $T \leq N + P$

$\Rightarrow \boxed{T = P + N}$

Theorem (2) if $a < c < b$ then $T_f[a, b] = T_f[a, c] + T_f[c, b]$

Proof : Consider any partition of $[a, b]$ and $t[a, b] = T_f[a, b]$

Add the point c to the Q partition, then t increases to t' say, and

$$t[a, b] \leq t'[a, c] + t'[c, b] \leq T[a, c] + T[c, b]$$

So we have $T[a, b] \leq [a, c] + T[c, b]$

Now take any partition of $[a, c]$ and $[c, b]$

gives $t[a, c]$ and $t[c, b]$ these partition gives a partition of $[a, b]$ and we see that

$t[a, c]+t[c, b] \le T[a, b]$ on taking supremum over all such pairs of

partitions gives $T[a, c]+T[c, b] \le T[a, b]$

$\Rightarrow \boxed{T[a, c]+T[c, b]=T[a, b]}$

Theorem (3) (Decomposition theorem for function of bbd variation)

A function $f \in BV[a, b]$ iff f is the difference of two finite valued montone increasing functions on $[a, b]$ where a and b finite.

Proof : Suppose that f is of bbd variation,

put $g(x)=p_f[a, x]+f(a)$ and $h(x)=N_f[a, x]$

then g and h are montone increasing functions and $0 \le p_f[a, x] \le T_f[a, x] \le T_f[a, b]$ so g and similarly h is finite.

But $f=g-h$ on $[a, b]$

<u>Conversly :</u>

Let $f=g-h$ where g and h are finite valued montone increasing functions then for any partition

$a=x_0<x_1<x_2, \ldots\ldots<x_n=b$ we have

$$\sum\left|f(x_i)-f(x_{i-1})\right| \le \sum\left(g(x_i)-g(x_{i-1})\right) + \sum\left(h(x_i)-h(x_{i--1})\right) \le g(b)-g(a)+h(b)-h(a)$$

So $T_f[a, b]<\infty$

$\Rightarrow f$ is of bbd variation on $[a, b]$

Assignment (1) prove that $BV[a, b]$ is a vector space :

Assignment (2) prove that $f(x)$ on $[0, 1]$ defined by

$$f(x)=\begin{cases} \sin\left(\dfrac{\pi}{x}\right); & x>0 \\ 0 \quad ; & x=0 \end{cases}$$ is not function of bbd variation.

Assignment (3) show that $g(x)$ on $[0, 1]$ defined by $g(x)=\begin{cases} x\sin\dfrac{\pi}{x}; & x>0 \\ 0 \quad : & x=0 \end{cases}$ is continuous but

$g \notin BV[0, 1]$

Assignment (4) prove that a function f of bounded derivative on $[a, b]$ is a function of bbd variation where f' is continuous on $[a, b]$.

LEBESGUE'S DIFFERENTIATION THEOREM

Theorem (1) : Let G be a finite collection of intervals $\left[I_K\right]$ then there exist a sub-collection G_0 of disjoint intervals of G, $G_0 = \left[I_{k_i}\right]$ say such that $m\left(\bigcup I_{k_i}\right) \geq \frac{1}{3} m\left(\bigcup I_k\right)$

Proof : Let $I_{K_1} \in G$ be an interval of maximal length. Remove from G any interval meeting I_{K_1} The measure of the union of these intervals (including I_k) is not greater than $3l\left(I_{K_1}\right)$ as $l\left(I_{K_1}\right)$ is maximal. This leaves a smaller. Class G_1 from which I_{K_2} is similarly chosen and the measure of the union of the intervals meeting I_{K_2} is not greater than $3l\left(I_K\right)$ etc. continue until G is exhausted to get intervals $I_{K_1}, I_{K_2}, \ldots. I_{K_n}$ which are disjoint from the construction. Every interval of G meets. Some I_{K_i} so

$$m\left(\bigcup I_k\right) \leq \sum 3l\left(I_{K_i}\right) = 3m\left(\bigcup_{i=1}^{n} I_{K_i}\right)$$

$$\Rightarrow \quad m\left(\bigcup_{i=1}^{n} I_{K_i}\right) \geq \frac{1}{3} m\left(\bigcup I_K\right)$$

Theorem (2) : If $\left[I_\alpha\right]$ is collection of open intervals such that $m\left(\bigcup I_\alpha\right) < \infty$ there exists a finite sub-collection $I_1, I_2, \ldots\ldots I_n$ of there intervals such that

$$m\left(\bigcup I_K\right) \geq \frac{1}{2} m\left(\bigcup I_\alpha\right)$$

Proof : By Lindelof theorem

We may choose a countable sub-collection $\left[I_K\right]$ of the $\left[I_\alpha\right]$ with the same union.

$$\lim m\left(\bigcup_{K=1}^{n} I_K\right) = m\left(\bigcup I_\alpha\right) < \infty$$

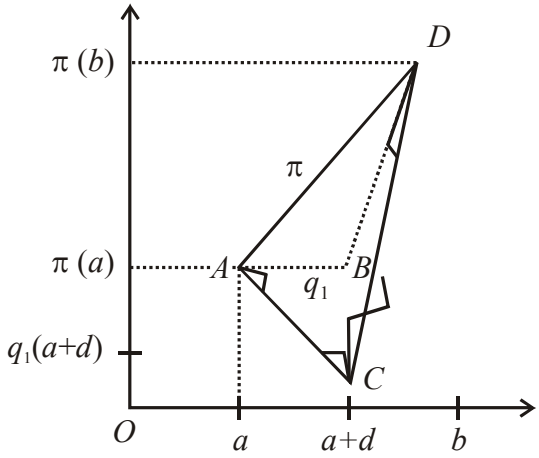
So n exist with the desired property.

Note : If $c < d$ and f is any function then $f(c, d)$ stands to ratio $\dfrac{f(d) - f(c)}{d - c}$

Theorem (3) Let $\pi(x)$ be a linear on $[a, b]$, $\pi(a) \leq \pi(b)$. Let q be a polygon with same end points as π of which n sides the total length of whose projections on the x-axis is d, have a slope less than $-\xi$; $(\xi > 0)$ then $\ell(q) > l(\pi) + d\left(\sqrt{\left(1 + \xi^2\right)} - 1\right)$.

Proof : Starting with q ... replace adjacent sides, where necessary, by moving them parallel to themselves until, after a finite number of steps, there is obtained a new polygon q_1 with sides congruent to those of q and whose first n sides have slope $< -\xi$. As each replacement leaves length unchanged $l(q) = l(q_1)$. Clearly $q_1(a, a + d) < -\xi$.

In aside figure;



B is the point $\left(a+d,\ \pi(a)\right)$;

C is the point $\left(a+d,\ q_1\left(a++d\right)\right)$,

Now $AC = AB$

$\sec < BAC > AB\sqrt{1+\xi^2}$

So $l(\pi) = AD < AB + BD < AB + CD < AB + CD + AC - AB\sqrt{1+\xi^2}$

But $AB = d$ and $CD + AC \le l(q_1) = l(q)$

So $\ell(\pi) < \ell(q) - d\left(\sqrt{1+\xi^2} - 1\right)$

Corollary : If π and q are defined as above (Theorem-3) but with $\pi(a) \ge \pi(b)$ and with n sides of q having a slope greater than $\pi(a) \ge \pi(b)$ then the sum conclusion holds.

Proof : To prove it replace π by $-\pi$, q by $-q$ and apply the theorem (3).

Theorem (4) (Lebesque's Differentiation of Theorem) :

If $f \in BV[a,\ b]$ where a and b are finite, then we have (i) f is differentiable a.e. (ii) the derivative is finite a.e.

Proof : (i) It is sufficient to show $D^+ f \le D_- f$ a.e. since $-f \in BV[a,\ b]$

We have $D_+ f \ge D^- f$ a.e. this gives $D^+ f \ge D_+ f \ge D^- f \ge D_- f \ge D^+ f$ and equality holds a.e.

We suppose that $D^+ f > D_- f$

on a set of positive measure and obtain a contradiction.

$\Rightarrow$ f is continuous. a.e.

$\Rightarrow$ the derivates are measurable.

Also there exists $\varepsilon > 0$ and a set $F \subseteq [a,\ b]$ with $m(F) > 0$ and such that $D^+ f - D_- f > 2\varepsilon$ on F but

$\left\{x : D^+ f - D_- f > 2\varepsilon\right\} = \cup\left\{x : D^+ f(x) > r_h + D_- f(x) < r_h - \varepsilon\right\}$

where $\{r_h\}$ is an enumeration of the rationals.

So at least one set of this union has positive measure. We can therefore find number ε, h with $\varepsilon > 0$ and a set E in $[a, b]$ with $m(E) > 0$ and on which f is continuous.

Such that $\quad D^+ f > n + \varepsilon_1$

$\quad D_- f < n - \varepsilon$ on E

Now $f - nx \in BV[a, b]$ and $D^+(f - nx) > D_-(f - nx)$

If and only if $D^+ f > D_- f$

So we may suppose that $n = 0$

Let π be any polygon drawn as in theorem (3) to approximate f and let p be the set of points of the corresponding partition of $[a, b]$ let $x \in E - P$ and suppose that $\pi'(x) < 0$.

$\because \quad D^+ f(x) > \varepsilon$ there exists $b_x > x$ such that $f(x, b_x) > \varepsilon$ then as f is continuous at x and hence $f(x, \beta)$ is continuous function of x we can find $a_x < x$ such that $f(a_x, b_x) > \varepsilon$ and clearly we may choose a_x and b_x so that π is linear on (a_x, b_x).

Similarly, if $\pi'(x) \geq 0$ we use that fact $D_- f < -\varepsilon$ and choose an interval $(a_x - b_x)$ on which π is linear and $f(a_x, b_x) < -\varepsilon$

then $\bigcup_x (a_x, b_x) \supseteq E - P$ so by theorem (2)

there exists a finite collection of these intervals say $I_1, I_2, \ldots. I_n$ such that

$$m\left(\bigcup_{k=1}^{h} I_k\right) > \frac{1}{2} m\left(\bigcup_x (a_x, b_x)\right) \geq \frac{1}{2} m(E - P) = \frac{1}{2} m(E)$$

By theorem (1)

We may extract a subcollection of disjoint intervals

$I_{K_1}, \ldots. I_{K_r}$ from these, such that

By theorem (3)

To each interval on which π is linear and adding we get

$$\ell(q) > \ell(\pi) + \sum_{i=1}^{r} \ell(I_{K_i})\left(\sqrt{1+\xi^2} - 1\right) > \ell(\pi) + \frac{1}{6} m(E)\left(\sqrt{1+\xi^2} - 1\right)$$

But ξ is independent of π so

Since $\ell(\pi)$ can always be increased by a constant amount. $\text{Supl}(\pi) = \infty$ taking the supermum over all poygon π approximating f: Hence $f \notin BV[a, b]$ and get contradiction.

$\Rightarrow$ (i) f is differentiable a.e.

Now,

(ii) Suppose this result is false, then replacing f by $-f$ if necessary we may suppose that there exists a set

E on which f is continuous, $E \subseteq [a, b]$ $m(E) > 0$ and $D^+ f = \infty$ on E. Then for any $M > 0$ choose. As (i) a collection of intervals $[(a_x, b_x)]$ covering E such that $f(a_x, b_x) > M$ choose the disjoint intervals $I_{K_1}, ...I_{K_r}$ as before such that $\sum_{k=1}^{r} \ell(I_{K_i}) > \frac{1}{6} m(E)$. Let q be the polygon, approximating f, determined by the end-points of the intervals I_{K_i} the length of q in the interval I_{K_i} is greater than $\ell(I_{K_i})\sqrt{1+M^2}$ since the slope of f is greater than M. So $\ell(q) > \sum_{i=1}^{r} \ell(I_{K_i})\sqrt{1+M^2} > \frac{1}{6} m(E)\sqrt{1+M^2}$

But M is arbitrary and E is independent of M so taking the supermum over all approximating polygons π we get $\sup \ell(\pi) = \infty$ and (ii) proved.