M.S c Mathematics –SEM 2 Number Theory, CC-10 , Unit 4

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Content: Infinite Continued Fractions

Infinite Continued Fractions

Definition: A continued fraction $< a_1 , a_2, ...a_n>$ havinf infinite number of partial quotients is called an infinite continued fraction and its value is defined to be equal to

$\lim_{n\to \infty} < a_1 , a_2, ...a_n> = \alpha$

So we write $\alpha = < a_1 , a_2, ...>$

Similarly for any positive integer k , $< a_1 , a_2, ...a_k >$ is called the k-th convergent of α .

Again , for any positive integer k $< a_k , a_{k+1}, ... >$

Is called the k-th complete quotient of $< a_1 , a_2, ... >$ or of α .It is denoted usually by α_k .

Here we study the properties of continued fractions having infinite number of partial quotients.

Theorem

Let a_n be a positive integer for every n > 0 except that a_1 may be zero. Then the continued fraction $< a_1 , a_2, ...a_n>$ converges to a finite limit as n tends to infinity.

Proof:

Consider the convergence of $< a_1 , a_2, ...a_n>$ ....(1)

To prove this , we have to prove two theorem that is ' the odd convergents form a strictly increasing sequence and the even convergent a strictly decreasing one' so

We have $\frac{Pn}{qn} - \frac{Pn-2}{qn-2} =$ **$(-1)^{n-1}a_n/q_nq_{n-2}$**

Where a_n, q_n , and q_{n-2} are all positive integers.

If n is odd , $(-1)^{n-1}=1$

Hence have $\frac{Pn}{qn} > \frac{Pn-2}{qn-2}$ **................ (i)**

If n is even , $(-1)^{n-1}= -1$ and

$\frac{Pn}{qn} < \frac{Pn-2}{qn-2}$ **........................ (ii)**

So from (i) and (ii) proves the theorem

Similarly again we prove, second theorem 'The value of a continued fraction is less than every even convergent , and greater than every odd convergent' to prove this we have

$$x-\frac{Pn}{qn} = \frac{Xn+1\ Pn + Pn-1}{Xn+1\ qn + qn-1} - \frac{Pn}{qn}$$

$$= (-1)\frac{Pn\ qn-1\ - Pn-1\ qn}{qn(Xn+1\ qn\ + qn-1)}$$

$$= (-1)^{n+1}/\ q_n\,(x_{n+1}\,q_n + q_{n-1})$$

Hence $x-\frac{Pn}{qn}$ **is negative when n is even and positive when n is odd. This proves the theorem.**

Now the main proof of the theorem that convergents of (1) form a strictly increasing sequence bu remain less than every even convergent. Thus $\frac{P2n-1}{q2n-1}$ increases as n n increases but is less than x-$\frac{P2}{q2}$. Letting n tends Type equation here.**to ∞ ,**

It follows that

$\mathrm{Lim}_{n}\rightarrow \infty\, P_{2n-1}/q_{2n-1} = \alpha_1$

Where α_1 is some positive real number $\leq P_2 / q_2$.

Similarly P_{2n}/q_{2n} decreases strictly as n increases but remains greater than P_1/q_1 .

Hence $\mathrm{Lim}_{n}\rightarrow \infty\, P_{2n}/q_{2n}= \alpha_2$

Where $\alpha_2 \geq P_1 /q_1$.

But $\alpha_2 - \alpha_1 = \mathrm{Lim}_{n}\rightarrow \infty\, (P_{2n}/q_{2n}{}^{---} P_{2n-1}/q_{2n-1})$

$= \lim_{n}\rightarrow \infty\, (1/q_{2n}\, q_{2n-1})$

$= 0$

Hence $\alpha_2 = \alpha_1 = \alpha$, say where $P_1 /q_1 < \alpha < P_2/q_2$. This implies that P_n /q_n or $< a_1 , a_2, \ldots\ldots a_n>$ converges to the value α as $n\rightarrow \infty$.

Theorem:

The value of an infinite continued fraction is irrational.

Let there exists a rational number say x such that

x= < a_1 , a_2, ... >. But we know that every rational number can be represented by a finite CF .

Hence x= < b_1 , b_2, ...b_N > for some integers b_1 , b_2, ...b_N .

It follows that < a_1 , a_2, ...> = < b_1 , b_2, ...b_N>.

We can then prove that

$\alpha_1 = b_1$, $\alpha_2=b_2$..................................$\alpha_{N-1}=b_{N-1}$

Leaving $a_N + 1/< a_{N+1} , a_{N+2}, ...a_n>$

= b_N

Which is impossible . Hence the theorem is true

Question

Find the CF for $\alpha=\frac{\sqrt{112}+8}{16}$

Solution : 16 divides $112-8^2$.

Hence α is of normal type

$$\alpha_1=\frac{\sqrt{112}+8}{16} = 1+\frac{\sqrt{112}-8}{16}$$

Therefore we obtain

$$\alpha_2=\frac{\sqrt{112}+8}{\left(\frac{112-8*8}{16}\right)} = \frac{\sqrt{112}+8}{3} = 6+\frac{\sqrt{112}-8}{3}$$

$$\alpha_3=\frac{\sqrt{112}+10}{4} = 5+\frac{\sqrt{112}-10}{4}$$

$$\alpha_4=\frac{\sqrt{112}+10}{3} = 6+\frac{\sqrt{112}-8}{3}$$

$$\alpha_5=\frac{\sqrt{112}+8}{16} = \alpha_1$$

Hence α = $\frac{\sqrt{112}+8}{16}$ = $<\overline{1, 6, 5, 6}>$ a purely periodic CF.

Assignment

(1) Express α = $\frac{\sqrt{37}+86}{33}$ as a CF.

(2) Find the CF representing $\sqrt{71}$.

(3)A periodic continued fraction represents a quadratic irrational.

(4)Find the value of $< 2, 4, \overline{1, 2, 3} >$