Measure Theory : (M.Sc. Sem-II)

By : Shailendra Pandit
Guest Assistant Prof. of Mathematics
P.G. Dept. Patna University, Patna

Email : sksuman1575@gmail.com
Call : 9430974625

UNIQUENESS OF THE EXTENSION

Notations :–

$\mathcal{S}(\mathbb{R})$: for the $\sigma-$ring $\mathcal{S}$ generated by the ring $\mathbb{R}$.

$\mathcal{K}\mathbb{R}$: for class consisting of $\mathcal{S}(\mathbb{R})$ together with all subsets of sets of $\mathcal{S}(\mathbb{R})$

μ^*: is an outer measure on $\mathcal{K}\mathbb{R}$ defined as

$$\mu^*(E) = \inf\left\{\sum \mu(E_n) : E_n \in \mathbb{R}, n \in N;\ E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}$$

$\mathcal{S}^*$ denotes sets of μ^* measurable sets.

Now using the definition of μ^*. We have extended the original measure μ on $\mathbb{R}$ to a complete measure $\overline{\mu}$ on $\mathcal{S}^*$ a $\sigma-$ring containing $\mathbb{R}$.

Theorem (1) : The outer measure μ^* on $\mathcal{K}(\mathbb{R})$ defined by μ on $\mathbb{R}$, and the corresponding outer measure defined by $\overline{\mu}$ on $\mathcal{S}(\mathbb{R})$ and $\overline{\mu}$ on $\mathcal{S}^*$ are same.

Proof : On observing that the outer measure β^* defined by a measure β on $\sigma-$ring $\mathbb{R}_1$ satisfied for $E \in \mathcal{K}(\mathbb{R}_1)$

$$\beta^*(E) = \inf\{\beta(F) : E \subseteq F \in \mathbb{R}_1\} \quad ...(e_1)$$

Since

$$\beta^*(E) = \inf\left\{\sum_{n=1}^{\infty} \beta(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathbb{R}_1\right\}$$

and replacing the sets E_n by disjoint sets $F_n \in \mathbb{R}_1$ such that $F_n \subseteq E_n$ and $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$ we get

$$\sum \beta(E_n) \geq \sum \beta(F_n) = \beta\left(\bigcup_{n=1}^{\infty} F_n\right) \geq \beta^*(E)$$

Thus (e_1) follows

Since $\mathcal{K}(\mathbb{R}) = \mathcal{K}(\mathcal{S}(\mathbb{R})) = \mathcal{K}(\mathcal{S}^*)$, the outer measures to be considered have the domain of definition as $\mu = \overline{\mu}$ on $\mathbb{R}$.

$$\mu^*(E) = \inf\left\{\sum_{n\geq 1} \mu(E_n) : E \subseteq \bigcup_{n\geq 1} E_n,\ E_n \in \mathbb{R}\right\}$$

$$= \inf\{\overline{\mu}(F) : E \subseteq F \in \mathcal{S}(\mathbb{R})\} \text{ by } (e_1)$$

$\geq \inf\left\{\bar{\mu}(F) : E \subseteq F \in \mathcal{S}^*\right\}$ as $\mathcal{S}^* \subseteq \mathcal{S}(\mathbb{R})$

$\geq \mu^*(E)$

Thus equality hold throughout and so by (e_1)

The outer measures are equal.

Note : Since the outer measure on $\mathcal{H}(\mathbb{R})$ determines the measurable sets and their measures, the measure and measurable sets obtained by extending. μ on $\mathbb{R}$, $\bar{\mu}$ on $\mathcal{S}(\mathbb{R})$ and $\bar{\mu}$ on $\mathcal{S}^*$ are the same namely on $\mathcal{S}^*$ without some restriction on $\bar{\mu}$ its extension to $\mathcal{S}(\mathbb{R})$ need not be unique, but we have.

Theorem (2) : If μ is a $\sigma-$finite measure on ring $\mathbb{R}$ then it has a unique extension to the $\sigma-$ring $\mathcal{S}(\mathbb{R})$.

Proof : Let $\bar{\mu}$ is an extension of μ on $\mathcal{S}(\mathbb{R})$ suppose υ is a measure on $\mathcal{S}(\mathbb{R})$ such that $\mu = \upsilon$ on $\mathbb{R}$. We wish to show that $\bar{\mu} = \upsilon$ on $\mathcal{S}(\mathbb{R})$. If $E \in \mathcal{S}\mathbb{R}$ and $\in > 0$, $\exists\{E_n\}$, $E_n \in \mathbb{R}$ $E \subseteq \bigcup\limits_{n\geq 1} E_n$ such that $\bar{\mu}(E) + \in \geq \sum\limits_{n\geq 1} \mu(E_n)$. But $A = \bigcup\limits_{n\geq 1} E_n$ may be written as disjoint.

Union of F_n, $F_n \subseteq E_n$, $F_n \in \mathbb{R}$ so we get $\bar{\mu}(E) + \in \geq \sum\limits_{n\geq 1} \mu(E_n) = \sum\limits_{n\geq 1} \upsilon(E_n) = \upsilon(A) \geq \upsilon(E)$

So $\bar{\mu}(E) \geq \upsilon(E)$

Suppose that $E \in \mathcal{S}(\mathbb{R})$, $\bar{\mu}(E) < \infty$ and $\in > 0$

then as above there exist $A \supseteq E$ such that $\bar{\mu}(A) < \bar{\mu}(E) + \in$ where $A = \bigcup\limits_{n\geq 1} F_n$ the sets F_n being disjoint sets of $\mathbb{R}$.

So that $\bar{\mu}(A) = \upsilon(A)$ so

$\bar{\mu}(E) \leq \bar{\mu}(A) = \upsilon(E) + \upsilon(A - E)$

But by the first part $\upsilon(A - E) \leq \bar{\mu}(A - E)$ also

Since $\bar{\mu}(E) < \infty$ we have $\bar{\mu}(A - E) < \in$

So $\bar{\mu}(E) \leq \upsilon(E) + \in$; Hence $\bar{\mu}(E) = \upsilon(E)$ if $\bar{\mu}(E) < \infty$ but

Now μ is $\sigma-$finite, for each $E \in \mathcal{S}(\mathbb{R})$ we have $E \subseteq \bigcup\limits_{n\geq 1} E_n$ where, for each $n_1 - E_n \in \mathbb{R}$ and $\mu(E_n) < \infty$ then we may write $E = \bigcup\limits_{n\geq 1} F_n$ where the F_n are disjoint sets of $\mathbb{R}$ and $\mu(F_n) < \infty$ so

$$\bar{\mu}(E) = \sum \mu(F_n) = \sum_{n\geq 1} \mu(F_n) = \upsilon(E) \text{ proved.}$$

SOME DEFINITIONS :

Definition (1) : A pair $[X, \mathcal{S}]$ where $\mathcal{S}$ is a $\sigma-$algebra of subsets of a space X, is called a measurable space the members of $\mathcal{S}$ are the measurable sets.

Definition (2) : A triplet $[X, \mathcal{S}, \mu]$ is called a measure space if $[X, \mathcal{S}]$ is a measurable space and μ is a measure on $\mathcal{S}$.