Complex Analysis : (M.Sc. Sem-II)

By : Dr. Loknath Rai

Prof. & Head of P.G. Dept. of

Mathematics, Patna University, Patna

Email : loknathrai7@gmail.com

Call : 9006632911

CONTOUR INTEGRATION CONTINUED

(1.1) Contour integration of trigonometric integrand.

(i) **Evaluate** $\int_{-\pi}^{\pi} \frac{1}{1+3\cos^2 t}\,dt$

Solution :– Put $z = e^{it}$

$$\frac{1}{z} = e^{-it}$$

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and $\frac{dz}{dy} = iz;\qquad dt = \frac{dz}{iz}$

Taking C to be the unit circle we get

$$\int_{-\pi}^{\pi} \frac{1}{1+3\cos^2 t}\,dt = \oint_C \frac{1}{1+\frac{3}{4}\left(z+\frac{1}{z}\right)^2}\frac{1}{iz}\,dz$$

$$= \oint_C \frac{-i}{z+\frac{3}{4}z\left(z^2+\frac{1}{z^2}+2\right)}\,dz$$

$$= -i\oint_C \frac{dz}{\frac{3}{4}z^3+\frac{5}{2}z+\frac{3}{4z}}$$

$$= -i\oint_C \frac{4}{3z^3+10z+\frac{3}{z}}\,dz$$

$$= -4i\oint_C \frac{z\,dz}{3z^4+10z^2+3}$$

$$= -\frac{4i}{3}\oint_C \frac{z\,dz}{\left(z+\sqrt{3}i\right)\left(z-\sqrt{3}i\right)\left(z+\frac{1}{\sqrt{3}}\right)\left(z-\frac{1}{\sqrt{3}}\right)}$$

The singularities to be considered are at $\frac{\pm i}{\sqrt{3}}$

Let C_1 be a small circle about $\frac{i}{\sqrt{3}}$ and C_2 be a small circle about $\frac{-i}{\sqrt{3}}$, then we get

$$= \frac{-4i}{3}\left[\oint_{C_1}\left(\frac{\dfrac{z}{(z^2+3)\left(z+\dfrac{i}{\sqrt{3}}\right)}}{z-\dfrac{i}{\sqrt{3}}}\right)dz + \oint_{C_2}\left(\frac{\dfrac{z}{(z^2+3)\left(z-\dfrac{i}{\sqrt{3}}\right)}}{z+\dfrac{i}{\sqrt{3}}}\right)dz\right]$$

$$= \frac{-4i}{3}\left[2\pi i\left(\frac{z}{(z^2+3)\left(z+\dfrac{i}{\sqrt{3}}\right)}\right)_{i=\frac{i}{\sqrt{3}}}\right] + 2\pi i\left[\left(\frac{z}{(z^2+3)\left(z-\dfrac{i}{\sqrt{3}}\right)}\right)_{i=\frac{-i}{\sqrt{3}}}\right]$$

$$= \frac{8\pi}{3}\left[\frac{\dfrac{1}{\sqrt{3}}}{\left(\dfrac{4}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)} + \frac{\dfrac{1}{\sqrt{3}}}{\left(\dfrac{2}{\sqrt{3}}\right)\left(\dfrac{4}{\sqrt{3}}\right)\left(\dfrac{2}{\sqrt{3}}\right)}\right] = \frac{8\pi}{3}\left[\frac{\dfrac{2}{\sqrt{3}}}{\dfrac{16}{3\sqrt{3}}}\right]$$

$$= \frac{8\pi}{3} \times \frac{3}{8} = \pi$$

(ii) **Evaluate** $I = \dfrac{1}{4}\displaystyle\int_0^{2\pi} \frac{1}{1+\sin^2 t}\,dt$

Solution :– On making substitution $z = e^{it}$

We get $I = \dfrac{1}{4}\displaystyle\oint_{|z|=1} \frac{4iz}{z^4-6z^2+1}\,dz$

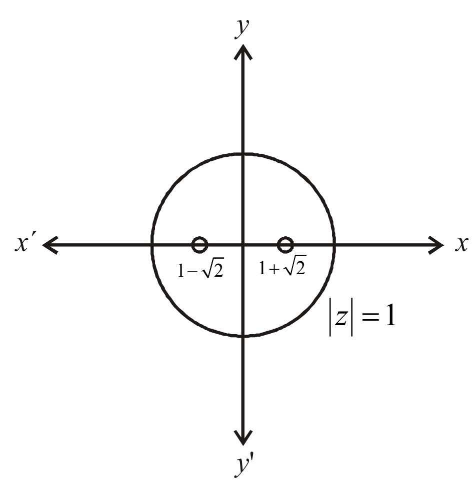
$$= \oint_{|z|=1} \frac{iz}{z^4-6z^2+1}\,dz$$

The poles of this function are at $1 \pm \sqrt{2}$ and $-1 \pm \sqrt{2}$ of these only $1-\sqrt{2}$ and $-1+\sqrt{2}$ are inside the unit circle.

Residues at $1+\sqrt{2}$

$$R\left(f, 1+\sqrt{2}\right) = \lim_{z \to 1+\sqrt{2}}\left(\frac{iz}{z^4-6z^2+1}\right)$$

$$= -\frac{i\sqrt{2}}{16}$$



Also

$$R\left(f, -1+\sqrt{2}\right) = \lim_{z \to \sqrt{2}-1}\left(\frac{iz}{z^4-6z^2+1}\right) = \frac{-i\sqrt{2}}{16}$$

Hence,

$$I = 2\pi i\left[\text{sum of residues}\right] = 2\pi i\left(-\frac{i\sqrt{2}}{16} - \frac{i\sqrt{2}}{16}\right)$$

$$= 2\pi i\left[-\frac{i\sqrt{2}}{8}\right] = \frac{\pi\sqrt{2}}{4}$$

$$= \frac{\pi}{2\sqrt{2}}$$

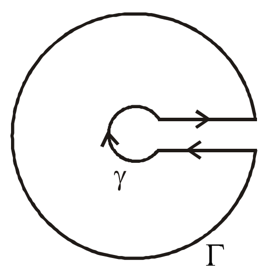
III. BRANCH CUTS

Evaluate : $\int_0^\infty \frac{\sqrt{x}}{x^2+6x+8}\,dx$

Solution :– We start by formulating the complex integral

$$I = \int_C \frac{\sqrt{z}}{z^2+6z+8}\,dz$$

Note that $z^{\frac{1}{2}} = e^{\frac{1}{2}\operatorname{Log} z}$ So has a branch cut this affects our choice of the Contour C.



Normally the logrithm branch cut is defined as the calculation of the integral slightly more complicated. so we define it to be the positive real axis. We use the keyhole contour which consists of a small circle about the origin of radius $\in$ (say) extending to a line segment parllel and close to the positive real axis but not touching it to an almost full circle returning to a line segment parallel close and below the positive real axis in the negative sense returning to the small circle in the middle.

Note that $z = -2$ and $z = -4$ are inside the big circle these are two remaining poles.

Let γ be the small circle of radius $\in$, T the larger, with radius R, then

$$\int_C f(z)\,dz = \int_{\in}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz + \int_{R}^{\in} f(z)\,dz + \int_{\gamma} f(z)\,dz$$

It can be shown that $\int_{\Gamma} f(z)\,dz$ & $\int_{\gamma} f(z)\,dz$

both going to zero as $\in \to 0$ and $R \to \infty$

$\because$ $z^{\frac{1}{2}} = e^{\frac{1}{2}\operatorname{Log} z}$ on the contour outside the branch cut, we have gained 2π in argument along γ.

$$\int_R^{\in} f(z)\,dz = \int_R^{\in} \frac{e^{\frac{1}{2}\operatorname{Log} z}}{z^2+6z+8}\,dz = \int_R^{\in} \frac{e^{\frac{1}{2}\left(\operatorname{Log}|z| + i\arg z\right)}}{z^2+6z+8}\,dz$$

$$= \int_R^{\in} \frac{e^{\frac{1}{2}\log|z|}\, e^{\frac{1}{2}(2\pi i)}}{z^2+6z+8}\,dz$$

$$= \int_R^{\in} \frac{e^{\frac{1}{z}\log|z|}\, e^{\pi i}}{z^2+6z+8}\,dz$$

$$= \int_R^{\in} \frac{-\sqrt{z}}{z^2 + 6z + 8}\, dz = \int_{\in}^{R} \frac{\sqrt{z}}{z^2 + 6z + 8}\, dz$$

$$\therefore \qquad \int_C \frac{\sqrt{z}}{z^2 + 6z + 8}\, dz = 2\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 6x + 8}\, dx$$

By using residue theorem

$$\int_0^{\infty} \frac{\sqrt{x}dx}{x^2 + 6x + 8} = \pi i\left(\frac{1}{\sqrt{2}} - i\right) = \pi\left(1 - \frac{1}{\sqrt{2}}\right)$$