M.S c Mathematics –SEM 2 Number Theory, CC-10 , Unit 4

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Content: Continued Fractions

Continued Fractions

Consider the following expression $3+\frac{1}{4+\frac{1}{6+\frac{1}{2}}}$ ....(1).

The value of this expression can be calculated by the ordinary arithmetical process

$6+\frac{1}{2}=\frac{13}{2}$,

$4+\frac{1}{6+\frac{1}{2}}=4+\frac{2}{13}=\frac{54}{13}$,

$3+\frac{1}{4+\frac{1}{6+\frac{1}{2}}}$ = $3+\frac{13}{54}=\frac{175}{54}$ which is the value of expression (1).

(1) is called a **simple continued fraction**. It is usually written in one of the following forms

$3+\frac{1}{4+}\frac{1}{6+}\frac{1}{2}$ or < 3, 4 , 6 ,2 >

Definition: Let a_1 be an arbitrary integer and a_2, a_3 .... a_N be positive integers. Then the expression

$a_1+\frac{1}{a2+}\frac{1}{a3+..}\frac{1}{aN}$....................................... (1) is called simple continued fraction. We may call (1) **a continued fraction or CF**. For convenience (1) is usually written as

$a_1+\frac{1}{a2+}\frac{1}{a3+..}\frac{1}{aN}$.................................. (2)

or as

is usually written as < a_1 ,a_2 , ... > ....(3).

The terms a_1, a_2, ...a_N of (1) are called partial quotients. a_1 is the first partial quotients .

Theorem

Every Continued fraction with finite number of partial quotients a_1 , a_2 , a_3 , ..., a_N represents a rational fraction.

Solution: We shall denote this rational fraction by the letter .

$$x = a_1 + \frac{1}{a2+}\frac{1}{a3+.}\frac{1}{aN}$$

$$x = < a_1, a_2, ...a_N >$$

and we say $< a_1, a_2, ...a_N >$ is the continued fraction of x or the continued fraction expression of x.

Every rational fraction can be expressed as a continue fraction with finite number of partial quotients. Suppose the given fraction is $\frac{103}{24}$.

Then $\frac{103}{24} = 4 + \frac{7}{24} = 4 + \frac{1}{\frac{24}{7}}$.

$\frac{24}{7} = 3 + \frac{3}{7} = 3 + \frac{1}{\frac{7}{3}}$

$\frac{7}{3} = 2 + \frac{1}{3}$

Hence $\frac{103}{24} = 4 + \frac{1}{3+}\frac{1}{2+}\frac{1}{3}$

$= < 4 , 3 , 2, 3 >$

We can carry out the above process in a simpler way as follows (similar to the process of finding the g. c .d. of 103 and 24

24 | 103 | 4

96

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7 | 24 | 3

21

—

3 | 7 |2

6

—

1 |3 | 3

3

_

0

Therefore $\frac{103}{24} = < 4, \ 3, \ 2, \ 3 >$.

Theorem

Every rational fraction can be expressed as simple continued fraction with the last quotient greater than 1 in a unique way.

Proof: Let $\frac{a}{b}$ be the given fraction .Then we form the following sequence of equations exactly as in the case of Euclid's algorithm for a and b.

$a = ba_1 + r_1$, $o < r_1 < b$ ....(1)

$b = r_1a_2 + r_2$, $o < r_2 < r_1$.........................(2)

$r_1 = r_2a_3 + r_3$, $o < r_3 < r_2$.........................(3)

..

..

$r_{N-2}=r_{N-1}a_N + r_N$, $0=r_N$.........................(N)

Since $r_1> r_2 > r_3>$ .... there exists an integer N such that $r_N=0$,and there the process ends. From equation (N) it is clear that $a_N> 1$.

Note if b>a , then $a_1=0$ and if $\frac{a}{b}$ is negative, a_1 is negative. From equation (1) to (N) we have

$\frac{a}{b}=a_1+ r_{1/b} =a_1 + \frac{1}{\frac{b}{r1}}$

....

....

$=a_1 + \frac{1}{a2+}\frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+aN}$................................. (A)

Thus $\frac{a}{b}$ is expressed as a simple continued fraction with $a_N > 1$ .WE have now to prove that the expansion A is unique.

Suppose it is not uniue. Then it follows that

$a_1 + \frac{1}{a2+}\frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+aN}= b_1 + \frac{1}{b2+}\frac{1}{b3+}\frac{1}{b4+.} \dots \dots \frac{1}{+bl}$ .... (B)

We take $l > N$, and $b_l > 1$ .Now in the above equation a_1 and b_1 are integers while

$\frac{1}{a2+}\frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+aN}$ and $\frac{1}{a2+}\frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+al}$

are rational proper fraction . Whence $a_1= b_1$ and

$\frac{1}{a2+}\frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+aN}= \frac{1}{b2+}\frac{1}{b3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+bl}$ which implies

$a_2 + \frac{1}{a3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+aN}=b_2 + \frac{1}{b3+}\frac{1}{a4+.} \dots \dots \dots . \frac{1}{+al}$......................(C)

Repeating the same argument with respect to equation (C) we obtain

$a_2 = b_2$ and $a_3 + \frac{1}{a4+.} \ldots \ldots \ldots . \frac{1}{+aN} = b_3 + \frac{1}{a4+.} \ldots \ldots \ldots . \frac{1}{+al}$.

In this way we obtain further in succession $a_3 = b_3$, $a_4 = b_4$, $a_{N-1}=b_{N-1}$

and are left with the equation

$a_N = b_N + (\frac{1}{bN+1+} \frac{1}{bN+2+} \qquad \frac{1}{+bl})$

Bu this is impossible since $l > N$ and $b_l>1$ .Hence N= l and $a_N=b_l$. It follows that the expansion (A) is unique.

Question : Find the continued fraction expansion of $\frac{21}{73}$.

Solution 73 | 21 | 0

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21 | 73 | 3

63

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10 | 21 |2

20

—

1 | 10 |10

10

—

00

Hence $\frac{21}{73} = < 0,3,2,10 >$

6

—

1 |3 | 3

3

–

0

Therefore $\frac{103}{24}$ = < 4, 3, 2, 3 > .