CONVERGENCE OF SEQUENCE OF REAL FUNCTIONS

(M.Sc. Sem-I)

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POINTWISE CONVERGENCE OF SEQUENCES OF FUNCTIONS

This chapter deals with sequences $\{f_n\}$ whose terms are real – or complex valued functions having a common domain on the real line R or in the complex plane C. For each *x* in the domain we can form another sequence $\{f_n(x)\}$ whose terms are the corresponding function values. Let *S* denote the set of *x* for which this second sequence converges. The function *f* defined by the equation

$$f(x) = \lim_{n\to\infty} f_n(x), \quad \text{if } x \in S,$$

Is called the limit function of the sequence $\{f_n\}$, and we say that $\{f_n\}$ converges pointwise to *f* on the set *S*.

Our chief interest in this chapter is the following type of question : If each function of a sequence $\{f_n\}$ has a certain property, such as continuity, differentiability, or integrability, to what extent is this property transferred to the limit function? For example, if each function f_n is continuous at *c*, is the limit function *f* also continuous at *c*? We shall see that, in general, it is not. In fact, we shall find that pointwise convergence is usually not strong enough to transfer any of the properties mentioned above from the individual terms f_n to the limit function *f*. Therefore we are led to study stronger methods of convergence that do preserve these properties. The most important of these is the notion of uniform convergence.

Before we introduce uniform convergence, let us formulate one of our basic questions in another way. When we ask whether continuity of each f_n at *c* implies continuity of the limit function *f* at *c*, we are really asking whether the equation

$$\lim_{x\to c} f_n(x) = f_n(c),$$

Implies the equation

$$\lim_{x\to c} f(x) = f(c).$$

But (1) can also be written as follows :

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x).$$

Therefore our question about continuity amounts to this : Can we interchange the limit symbols in (2)? We shall see that, in general, we cannot. First of all, the limit in (1) may not exist. Secondly, even if it does exist, it need not be equal to $f(c)$. We encountered a similar situation in Chapter 8 in connection with iterated series when we found that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m, n)$ is not necessarily equal to $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m, n)$.

The general question of whether we can reverse the order of two limit processes arises again and again in mathematical analysis. We shall find that uniform convergence is a far-reaching sufficient condition for the validity of interchanging certain limits, but it does not provide the complete answer to the question. We shall encounter examples in which the order of two limits can be interchanged although the sequence is not uniformly convergent.

EXAMPLES OF SEQUENCES OF REAL-VALUED FUNCTIONS

The following examples illustrate some of the possibilities that might arise when we form the limit function of a sequence of real-valued functions.

Example-1 : A sequence of continuous functions with a discontinuous limit function. Let $f_n(x) = x^{2n}/\left(1+x^{2n}\right)$ if $x \in R,\ n = 1,\ 2,\ ....$ The graphs of a few terms are shown in figure. In this case $\lim\limits_{n\to\infty} f_n(x)$ exists for every real *x*, and the limit function *f* is given by

$$f(x) = \begin{cases} 0 & if \quad |x| < 1, \\ \dfrac{1}{2} & if \quad |x| = 1, \\ 1 & if \quad |x| > 1. \end{cases}$$

Each f_n is continuous on R, but *f* is discontinuous at $x = 1$ and $x = -1$.

Examples-2 : A sequence of functions for which $\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim\limits_{n\to\infty} f_n(x)\,dx.$ Let $f_n(x) = n^2 x(1-x)^n$ if $x \in R,\ n = 1,\ 2,\ ....$ If $0 \le x \le 1$ the limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists and equals 0. Hence $\int_0^1 f(x)\,dx = 0$. But

$$\int_0^1 f_n(x)\,dx = n^2\int_0^1 x(1-x)^n\,dx$$

$$= n^2\int_0^1 (1-t)t^n dt = \frac{n^2}{n+1} - \frac{n^2}{n+2} = \frac{n^2}{(n+1)(n+2)}$$

So $\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx = 1.$ In other words, the limit of the integrals is not equal to the integral of the limit function. Therefore the operations of “limit” and “integration” cannot always be interchanged.

Example-3 : A sequence of differentiable functions $\{f_n\}$ with limit 0 for which $\{f_n'\}$ diverges. Let $f_n(x) = (\sin nx)/\sqrt{n}$ if $x \in R,\ n = 1, 2, \ldots.$ Then $\lim_{n\to\infty} f_n(x) = 0$ for every *x*. But $f_n'(x) = \sqrt{n}\cos nx,$ so $\lim_{n\to\infty} f_n'(x)$ does not exist for any *x*.

DEFINITION OF UNIFORM CONVERGENCE

Let $\{f_n\}$ be a sequence of functions which converges pointwise on a set $S[0, 1]$ limit function *f*. This means that for each point *x* in *S* and for each $\varepsilon > 0,$ there exists as *N* (depending on both *x* and ε) such that

$$n > N \quad \text{implies} \qquad |f_n(x) - f(x)| < \varepsilon$$

If the same *N* words equally well for every point in *S*, the convergence is said to be uniform on *S*. That is, we have

Definition 1 : A sequence of function $\{f_n\}$ is said to converge uniformly to *f* on a set *S* if, for every $\varepsilon > 0,$ there exists an *N* (depending only on ε) such that $n > N$ implies.

$$|f_n(x) - f(x)| < \varepsilon, \quad \text{for every } x \text{ in } S.$$

We denote this symbolically by writing

$$f_n \to f \text{ uniformly on } S.$$

When each term of the sequence $\{f_n\}$ is real-valued, there is a useful geometric interpretation of uniform convergence. The inequality $|f_n(x) - f(x)| < \varepsilon$ is then equivalent to the two inequalities

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \ldots(3)$$

If (3) is to hold for all $n > N$ and for all *x* in *S*, this means that the entire graph of f_n (that is, the set $\{(x, y) : y = f_n(x),\ x \in S\}$) lies within a "band" of height 2ε situated symmetrically about the graph of *f*.

A sequence $\{f_n\}$ is said to be uniformly bounded on *S* if there exists a constant $M > 0$ such that $|f_n(x)| \le M$ for all *x* in *S* and all *n*. The number *M* is called a uniform bound for $\{f_n\}$. If each individual function is bounded and if $f_n \to f$ uniform on *S*, then it is easy to prove that $\{f_n\}$ is uniformly bounded on *S*. (See exercise-1) This observation often enables us to conclude that a sequence is not uniformly convergent. For instance, a glance at figure tells us at once that the sequence of Example 2 cannot converge uniformly on any subset containing a neighbourhood of the origin. However, the convergence in this example is uniform on every compact subinterval not containing the origin.

UNIFORM CONVERGENCE AND CONTINUITY

Theorem 2. Assume that $f_n \to f$ uniformly on *S*. If each f_n is continuous at a point *c* of *S*, then the limit function *f* is also continuous at *c*.

Note : If *c* is an accumulation point of *S*, the conclusion implies that

$$\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$$

Proof : If *c* is an isolated point of *S*, then *f* is automatically continuous at *c*. Suppose, then, that *c* is an accumulation point of *S*. By hypothesis, for every $\varepsilon > 0$ there is an *M* such that $n \geq M$ implies.

$$\left| f_n(x) - f(x) < \frac{\varepsilon}{3} \right| \text{ for every } x \text{ in } S.$$

Since f_M is continuous at *c*, there is a neighbourhood *B*(*c*) such that $x \in B(c) \cap S$ implies.

$$\left| f_M(x) - f_M(c) \right| < \frac{\varepsilon}{3}.$$

But

$$\left| f(x) - f(c) \right| \leq \left| f(x) - f_M(x) \right| + \left| f_M(x) - f_M(c) \right| + \left| f_M(c) - f(c) \right|.$$

If $x \in B(c) \cap S,$ each term on the right is less than $\varepsilon/3$ and hence $\left| f(x) - f(c) \right| < \varepsilon.$ This proves the theorem.

Note : Uniform convergence of $\{f_n\}$ is sufficient but not necessary to transmit continuity from the individual terms to the limit function. In Example 2, we have a non-uniformly convergent sequence of continuous functions with a continuous limit function.