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e- content –M. Sc –Semester –III

Number Theory –CC-10

Infinite Continued fraction

A continued fraction $< a_{1,} a_2, \dots \dots . a_n >$ having infinite number of partial quotient is called an infinite continued fraction and its value is defined to be equal to

$$\lim_{n\to\infty} < a_{1,} a_2, \dots \dots \dots a_n >= \alpha$$

Note: (i)For any positive integer k, $< a_{1,} a_2, \dots \dots . a_k >$ is called the k convergent of α

(ii) Again for any positive integer k $< a_k , a_{k+1}, \dots \dots . . >$ is called the $< a_{1,} a_2, \dots \dots . a_n >$ or of α . It is usually denoted by α_k.

Theorem

Let $< a_1, a_2, \dots . . >= \alpha$ then

(i) $\alpha_n = a_n + \frac{1}{\alpha_{n+1}} =< a_n, \alpha_{n+1} >$

(ii) $[\alpha_n] = a_n$

Proof: Let n be the given integer

(i) $\alpha_n =< a_n , a_{n+1}, \dots \dots >$

$$= \lim_{m\to\infty} < a_n , a_{n+1}, \dots \dots a_m >$$

$$= \lim_{m\to\infty} a_n + \frac{1}{< a_{n+1,} a_{n+2, \dots\dots\dots\dots,} a_m >}$$

$$= a_n + \frac{1}{< a_{n+1,} a_{n+2, \dots\dots\dots\dots,} a_m >}$$

$$= a_n + \frac{1}{\alpha_{n+1}}$$

$$=< a_n, \alpha_{n+1} >$$

(ii) $\boldsymbol{\alpha_n = a_n + \frac{1}{\alpha_{n+1}}}$ **and**

$$\alpha_{n+1} = a_{n+1} + \frac{1}{\alpha_{n+2}}$$

$$Hence$$

$0 < \frac{1}{\alpha_{n+1}} < 1$

So it follows that $[\alpha_n] = a_n$

Corollary 1 $[\alpha] = a_1$

Corollary2 $\alpha_n > 1$ for every n

Corollary 3

$\alpha = < a_1, \alpha_2 >$

. $=$ "

$=< a_1, a_2, \dots \dots . a_{n,} \alpha_{n+1} >$

Theorem

The continued fraction representing an irrational number is infinite, and is also unique.

Proof:

Let α be a given irrational number. Then

$\alpha = \alpha_1 = [\,\alpha_1] + (\alpha_1 - [\alpha_1])$

$where$ $[\alpha_1]$ is an integer number and $\alpha_1 - [\alpha_1]$ is a positive irrational no < 1

We put $a_1 = [\alpha_1]$ and $[\alpha_2] = \frac{1}{\alpha_1 - [\alpha_1]}$

Then we have $\alpha_1 = a_1 + \frac{1}{\alpha_2}$

Where α_2 is an irrational no >1.

Repeating the same process with α_2 in place of α_1.

We get

$$\alpha_2 = a_2 + \frac{1}{\alpha_3}$$

$$where\ a_2 = [\alpha_2]$$

$[\alpha_3] = \frac{1}{\alpha_2 - [\alpha_2]}$

$$= an\ irrational\ number > 1$$

Then we get in succession

$$\alpha_1 = a_1 + \frac{1}{\alpha_2} >$$

. $$\alpha_2 = a_2 + \frac{1}{\alpha_3}$$

"

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}} >$$

where α_{n+1} is an irrational number > 1

The Process obviously never ends

Therefore the quotients a_1, a_2 ....are infinite in number.

And we obtain

$$\alpha = \alpha_1 =< a_1, a_2, \ldots \ldots . a_{n,} \alpha_{n+1} >$$

We have to still to show that

$$< a_1, a_2 \ldots \ldots \ldots \ldots \ldots . . a_{n,} a_{n+1} >$$

We have to still to show that

$< a_1, a_2 \ldots \ldots \ldots \ldots \ldots . . a_{n,} >= \frac{P_n}{q_n}$

Actually converges to the value α as $n \to \infty$

This is done as follows

$$\alpha = \alpha_1 =< a_1, a_2, \ldots \ldots . a_{n,} \alpha_{n+1} >$$

$$= \alpha_{n+1}$$

$$= \frac{\alpha_{n+1} P_n + P_{n-1}}{\alpha_{n+1} q_{n+} q_{n-1}}$$

$$\alpha - \frac{P_n}{q_n} = \frac{P_{n-1} q_n - P_n q_{n-1}}{q_n(\alpha_{n+1} q_n + q_{n-1})}$$

$$= \frac{(-1)^{n-1}}{q_n(\alpha_{n+1} q_n + q_{n-1})}$$

Which tends to zero as n tends to infinity. This means

$$\alpha = \lim_{n \to \infty} \frac{P_n}{q_n} = \lim_{n \to \infty} < a_1, a_2, \ldots \ldots . a_{n,} >$$

So $\alpha \;\; =< a_1, a_2, \ldots \ldots . a_{n,} >$.............(i)

This proves the first part of the theorem.
We now prove that the CF (1) is unique.

Suppose this is not true .It follows that

$$\alpha =< a_1, a_2, \dots \dots . >$$
$$=< b_1, b_2, \dots \dots . >$$

We have prove that $b_n = a_n$ for every n.
This implies that the two continued fraction expansions are identical.

Hence Proved