e-content (lecture-32)

by

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MATH SEM-3 CC-11 UNIT-1 (Functional Analysis)

Topic: Problem in Banach space .

Problem: let $B(E)$ denotes thet set of all operators on a normed linear space E. Show that in the algebra $B(E)$

Multiplication is related to the norm by

$\|T_1T_2\| \leq \|T_1\|\|T_2\|$ .Also prove that multiplication in $B(E)$ is jointly continuous .

Solution: since for all $x \in E$ we have

$$\|(T_1T_2)x\| = \|T_1(T_2x)\| \leq \|T_1\|\|T_2x\| \leq \|T_1\|\|T_2\|\|x\|$$

$$\|(T_1T_2)x\|\frac{1}{\|x\|} \leq \|T_1\|\|T_2\| \;\; for\ all\ x$$

$$\Rightarrow \sup\{ \|(T_1T_2)x\|\frac{1}{\|x\|} : \; x \in E , x \neq 0\} \leq \|T_1\|\|T_2\|$$

$$\|T_1T_2\| \leq \|T_1\|\|T_2\| .$$

Now we have to show that multiplication in $B(E)$ is jointly continuous.

Let $A_n \to A, T_n \to T \Rightarrow A_n\, T_n \to AT$

Now $\| A_n\, T_n - AT \| = \| A_n\, (T_n - T) + (A_n - A)T \|$

$$\leq \| A_n \| \| (T_n - T) \| + \|(A_n - A)\| \|T \|$$

$$\to \| A_n \| .0 + 0. \|T \| = 0$$

Hence $A_n\, T_n \to AT$.

So multiplication in $B(E)$ is jointly continuous.

Question: If M is a closed linear subspace of a normed linear space E , and if T is the natural mapping of E onto the quotient space E/M defined by $T(x) = x + M$

Show that t is a continuous linear transformation for which $\|T\| \leq 1$.

Solution : since for all $x, y \in E$ and a, b are scalar we have $T(ax + by) = (ax + by) + M$

$$= (ax + M) + (by + M)$$

$$= a(x + M) + b(y + M)$$

$$= aT(x) + bT(y)$$

So T is a linear transformation from E to E/M.

Also we have $\|T(x)\| = \|x + M\|$

$$= \inf\{\|\mathrm{x} + \mathrm{v}\| : \mathrm{v} \in \mathrm{M}\}$$

$$\leq \|x\| = 1. \|x\|$$

So T is continuous and

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in E, x \neq 0\right\} \leq 1.$$

END.