**e-content (lecture-30)**

by

DR ABHAY KUMAR (Guest Faculty)

P.G. Department of Mathematics

Patna University Patna

MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Problem in Hilbert space

Question: show that in a Hilbert space $H$an orthonormal set $S$ is complete iff $x \perp S \Rightarrow x = 0$.

Solution: suppose that an orthonormal set $S$ is complete and $x \perp S$.

Then to prove that $x = 0$ Let $x \neq 0$. Then $e = \frac{x}{\|x\|}$ is a unit vector such that $e \perp S$.

So $\{e, S\}$ is an orthonormal set properly containing $S$.

This contradicts that $S$ is complete orthonormal set.

Hence we must have $x = 0$.

Conversely suppose that $x \perp S \Rightarrow x = 0$. then we have to prove that $S$ is complete orthonormal set.

Suppose $S$ is not complete orthonormal set.

Then there exists a unit vector $e$ such that $\{e, S\}$ is an orthonormal set so we have $e \perp S$ then by hypothesis

$e = 0$ this is a contradiction since $e$ is a unit vector >

Hence $S$ must be complete .

Problem: If $\{e_i\}$ is an orthonormal set in a Hilberty space $H$ ,and if $x$ ,y are arbitrary vectors in $H$ then

$\sum_i |(x, e_i)\overline{(y, e_i)}| \leq \|x\|.\|y\|$

Solution: Let $S = \{e_i : (x, e_i)\overline{(y, e_i)} \neq 0\}$

Then $S$ is either empty or countable . if $S$ is empty then we have $(x, e_i)\overline{(y, e_i)} = 0$ $for\ all\ i$.

In this case we $0 = \sum_i |(x, e_i)\overline{(y, e_i)}| \leq \|x\|.\|y\|$ .

if $S$ is not empty Then $S$ is either finite or countably infinite . If $S$ is finite .then we can write

$S = \{e_1, e_2 \dots . e_n\}$ in this case

$\sum_i |(x, e_i)\overline{(y, e_i)}| = \sum_{i=1}^{n} |(x, e_i)\overline{(y, e_i)}| \leq \|x\|.\|y\|$ ....(1)

[ using Cauchy and Bessel inequality]

If $S$ is countably infinite . then we can write

$S = \{e_1, e_2 \dots . e_n \dots .\}$ in this case

$\sum_i |(x, e_i)\overline{(y, e_i)}| = \sum_{i=1}^{n} |(x, e_i)\overline{(y, e_i)}|$

Since the inequality (1) is true for all n.

So it must be true in the limit hence we have

$\sum_{i=1}^{\infty} |(x, e_i)\overline{(y, e_i)}| \leq \|x\|.\|y\|$ ....(2)

From (2) we get $\sum_{i=1}^{\infty} |(x, e_i)\overline{(y, e_i)}|$ is convergent

And so it is absolutely convergent .

Therefore $\sum_{i=1}^{\infty} |(x, e_i)\overline{(y, e_i)}| \leq \|x\|.\|y\|$ .

END.