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SEMESTER – II

C C – 09 (Topology)

Topic : Urysohn's lemma

<u>Theorem :</u> **(Urysohn's lemma) : If A and B are disjoint closed subsets of a normal space X. Then there exists a continuous real function $\boldsymbol{f}$ from X into [0, 1] such that $\boldsymbol{f(A) = \{0\}}$ and $\boldsymbol{f(B) = \{1\}}$.**

Proof : Since A and B are disjoint, then $A \cap B = \emptyset$, So $A \subseteq B^c$. . Since B is closed, Hence B^c is an open set containing the closed set A, So there exists an open set $G_{\frac{1}{2}}$ such that $A \subseteq G_{\frac{1}{2}} \subseteq \bar{G}_{\frac{1}{2}} \subseteq B^c$ Now $\bar{G}_{\frac{1}{2}}$ is an open set containing the closed set A and B^c and B^c is an open set containing the closed set $\bar{G}_{\frac{1}{2}}$ So there exists open sets $G_{\frac{1}{4}}$, $G_{\frac{3}{4}}$ such that

$$A \subseteq G_{\frac{1}{4}} \subseteq \bar{G}_{\frac{1}{4}} \subseteq G_{\frac{1}{2}} \subseteq \bar{G}_{\frac{1}{2}} \subseteq G_{\frac{3}{4}} \subseteq \bar{G}_{\frac{3}{4}} \subseteq B^c$$

continuing this process for each dyadic rational number of the form $l = \frac{m}{2^n}$, (where n = 1, 2, 3 ……… and m = 1, 2, …., $2^n - 1$) of [0, 1] we obtain an open set of the form G such that $r, s \in [0,1]$ with $r < s \Rightarrow A \subseteq G_r \subseteq \bar{G}_r \subseteq G_s \subseteq \bar{G}_s \subseteq B^c$, Let D be the set of all dyadic rational numbers in [0,1]

Now we define a function $f : X \to [0, 1]$

$$\text{by } f(x) = 1 \text{ if } x \in B$$

$$= \inf \{r \in D : x \in G_r\} \text{ if } x \notin B.$$

if $x \in A$ then $x \in G_r$ for all $r \in D$

Hence, $f(x) = \inf \ D = 0$

Therefore, $f(x) = 0$ if $x \in A$

$= 1$ if $x \in B$

Thus $f(A) = \{0\}$ and $f(B) = \{1\}$

It remains to prove that f is continuous on X.

Since the intervals [0, a[and] b, 1] (where 0 < a, b <1) form an open subbase for the subspace [0, 1] of the real line R.

It is sufficient to prove that

$$f^{-1}([0, a\,[\,) \text{ and } f^{-1}(\,]b, 1]\,) \text{ are open sets in X}$$

We have,

$0 \leq f(x) < a$ iff $x \in G_r$ for some $r < a$

For if $x \in G_r$ for some $r < a$, Then by the definition of infimum there exists some $r \in D$, such that $f(x) \leq r \leq a$ from which $x \in G_r$

Thus $f^{-1}([0, a[\,) = \{x \in G : 0 \leq f(x) < a\}$

$$= U\,\{G_r : r \in D, r < a\}$$

Thus $f^{-1}([\,0, a\,[\,)$ is open in X

Now we have to show that $f^{-1}(\,]b, 1\,])$ is open in X

Since $f^{-1}(\,]b, 1\,]) = f^{-1}([0,\ b]^c)$

Now, We have,

$$0 \leq f(x) \leq b \ \text{ iff } x \in G_r \ \text{ for all } \ r > b.$$

For if $x \in G_r$ for all $r > b$

Then $f(x) \leq r$ for all $r > b$, Then $f(x) \leq b$

Also if $f(x) \leq b$ Then $f(x) < r$

Hence

$$f^{-1}(\,[0, b\,]\,) = \{x \in X : 0 \leq f(x) \leq b\}$$

$$= \cap\ \{G_r :\ r \in D,\ r > b\} \quad \text{......... (i)}$$

Now for every $r > b$, there exists an $S \in D$ such that $r > s > b$ and hence $G_r \subseteq \bar{G}_s$

Consequently,

$$\cap\ \{G_r :\ r \in D, r > b\} = \cap\{\bar{G}_s :\ s \in D,\ s > b\} \quad \text{........... (ii)}$$

$$f^{-1}(\,]\,b, 1\,]) = [\cap\ \{\bar{G}_s : s \in D, s > b\}]^c \qquad \text{(from (i) and (ii))}$$

$$= \cup\ \{\bar{G}_s^c : s \in D,\ s > b\}$$

Hence, $f^{-1}(]\,b, 1])$ is an open set in X

It proves

Theorem : If A and B are disjoint closed subsets of a normal space X and [a, b] be any closed and bounded interval on the real line R. Then, there exists a continuous function $\boldsymbol{f} : \boldsymbol{X} \rightarrow [\boldsymbol{a}, \boldsymbol{b}]$ such that $\boldsymbol{f}(\boldsymbol{A}) = \{\boldsymbol{a}\}$ and $\boldsymbol{f}(\boldsymbol{B}) = \{\boldsymbol{b}\}$

Proof : If $a = b$, then we define a function f on X by $f(x) = a$ for all $x \in X$, then the results holds.

If $a < b$, then by Urysohn's lemma, there exists a continuous real function $g : X \rightarrow [0, 1]$ such that $g(A) = \{0\}$ and $g(B) = \{1\}$

Now we define a function f on X

by $f(x) = (b - a)\, g(x) + a$ for all $x \in X$ ……. (i)

Since $0 \leq g(x) \leq 1$

$$\Rightarrow (b - a).0 \leq (b - a).g(x) \leq (b - a).1$$

$$\Rightarrow 0 \leq (b - a)\, g(x) \leq b - a$$

$$\Rightarrow 0 + a \leq (b - a)\, g(x) + a \leq b - a + a$$

$$\Rightarrow a \leq f(x) \leq b$$

Thus f is a function from X to $[a, b]$

Again, since g is continuous so f is also continuous on X.

Now if $x \in A \Rightarrow g(x) = 0$

$\Rightarrow f(x) = a$ [from (i)]

if $x \in B \Rightarrow g(x) = 1$

$\Rightarrow f(x) = b$ [from (i)]

Thus $f(A) = \{a\}$ and $f(B) = \{b\}$

It proves.