M.S c Mathematics –SEM 3 Functional Analysis- CC-11

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Theorem

Let M be a closed linear subspace of a normed linear space E. If the norm of a coset [x]= x + M in the quotient space E/M is defined by

$$||x+M|| = inf\ \{||x+v||: v \in M\}$$

Then E/M is a normed linear space.Further,if E is a Banach space ,then E/M is also a banach space.

Proof

Clearly $||x + M\ ||$ is well-defined and $||x + M|| \geq 0$.

Since $0 \in M$,it is clear that if $x + M = M$, the zeros vector in E/M then

$$||x + M|| = ||M|| = \inf_{v \in M} ||v|| = 0$$

If $||x + M|| = 0$, then from the definition of the norm as an infimum, there exists a sequence (z_n) of points of M such that

$$\lim_{n \to \infty} ||x + z_n|| = ||x + M|| = 0$$

Hence $x + z_n \to 0$ or $z_n \to -x$, Since M is closed, $-x \in M$, hence $x \in M$ and therefore ,$x + M = M$, the zero vector of E/M.

Thus $||x + M|| = 0$ iff $x\ + M\ = M$, the zero vector of E/M. Next ,let a be any scalar. Then

$$||ax+M|| = \inf_{v\in M} ||ax+v|| = \inf_{v\in M} ||ax+av||$$

Provided $a \neq 0$ (the result being trivially obtained when $a=0$)

Hence

$$||ax+M|| = |a| \inf_{v\in M} ||x+v|| = |a|.||x+M||$$

Thus $||a[x]|| = |a|.||[x]||$

Finally, let $x+M.Y+M \in E/M$ .Then

$$(x+M)+(y+M) = (x+y)+M$$

$$||(x+M)+(y+M)|| = ||x+y+M||$$

But since $||x+y+M||$ has been defined as an infimum, so there exist sequences (z_n) and (w_n) of points of M such that |

$$\lim_{n\to\infty} ||x+z_n|| = ||x+M||$$

And $\lim_{n\to\infty} ||y+w_n|| = ||y+M||$

Thus $||[x] + [y]|| \leq ||[x]|| + ||[y]||$

Therefore E/M is a normed linear space.

Now let E be a Banach space . Then we prove that E/M is a Banach space

For this we start with a Cauchy sequence,in E/M.

Since it is assumed to be a Cauchy sequence, if a convergent subsequence of the sequence can be extracted, it follows that the entire sequence must converge to the same limit as the subsequence. It is clearly possible to find a subsequence $(\boldsymbol{x_n + M})$ **of the original Cauchy sequence such that**

$$||(\boldsymbol{x_1 + M}) - (\boldsymbol{x_2 + M})|| < \frac{\mathbf{1}}{\mathbf{2}}$$

$||(\boldsymbol{x_2 + M}) - (\boldsymbol{x_3 + M})|| < \frac{\mathbf{1}}{\mathbf{4}}$, **and in general**

$$||(\boldsymbol{x_n + M}) - (\boldsymbol{x_{n+1} + M})|| < \frac{\mathbf{1}}{\mathbf{2^n}}$$

We prove that this sequence is convergent in E/M.

We take any $y_1 \in x_1 + M$ and $y_2 \in x_2 + M$ such that $||y_1 - y_2|| < \frac{1}{2}$.

Again we select y_3 in $x_3 + M$ such that

$$||y_n - y_{n+1}|| < \frac{1}{2^n}$$

If $< n$, then

$$\begin{aligned} ||y_m - y_n|| \\ &\leq ||y_m - y_{m+1}|| + ||y_{m+1} - y_{m+2}|| \\ &+ \cdots \ldots \ldots . . ||y_{n-1} - y_n|| \\ &< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots \ldots . . + \frac{1}{2^{n-1}} \\ &< \frac{1}{2^{m-1}} \end{aligned}$$

Hence (y_n) is a Cauchy sequence in E. E is complete ,there exists $y \in E$ such that $y_n \to y$.

Now $||(x_n + M) - (y + M)|| \leq |y_n - y|$

Hence $x_n + M \to y + M \in E/M$

Therefore , E/M is complete. Hence $\frac{E}{M}$ is a Banach space.

Hence