e-content (lecture-26)

by

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Problem on projections

Theorem: If P and Q are the projections on closed linear subspaces M and N of H. Then prove that $Q-P$ is a projection $\Leftrightarrow P \leq Q$. In this case show that $Q-P$ is the projection on $N \cap M^{\perp}$.

Proof: Since P and Q are the projections on the Hilbert space H. So $P^2 = P, P^* = P, Q^2 = Q, Q^* = Q$.

Now suppose that $Q-P$ is a projection on H.

To prove $P \leq Q$.

Since $Q-P$ is a projection on $H. So\ Q-P$ is a positive operator on H .

so we have $Q - P \geq 0 \Rightarrow P \leq Q$.

Conversely suppose that $P \leq Q$.

To prove $Q - P$ is a projection on H.

$$(Q - P)^* = Q^* - P^* = Q - P$$

And $\quad (Q - P)^2 = (Q - P)(Q - P)$

$$= Q^2 - QP - PQ + P^2$$

$$= Q - P - P + P$$

$$= Q - P$$

Hence $Q - P$ is a projection on H.

It remains to prove that $Q - P$ is the projection on $N \cap M^{\perp}$.

i.e the range of $Q - P \ is \ N \cap M^{\perp}$.

Let R be the range of $Q - P$.To prove that $R = N \cap M^{\perp}$.

Let $x \in N \cap M^{\perp} \Rightarrow x \in N and \ x \in M^{\perp}$

Now $(Q - P)(x) = Qx - Px$

$$= x - Px$$

$$= x - 0 = x$$

$$\Rightarrow x \in R. \text{ [Since } (Q-P)(x) \in R\text{]}$$

So $N \cap M^{\perp} \subseteq R$ **........(1)**

Again let $x \in R$ then $(Q-P)(x) = x$

$$\Rightarrow Q[(Q-P)]x = Qx$$

$$\Rightarrow (Q^2)x - QPx = Qx$$

$$\Rightarrow Qx - Px = Qx$$

$$\Rightarrow (Q-P)(x) = Qx$$

$$\Rightarrow x = Qx$$

$$\Rightarrow x \in N.$$

Again $(Q-P)(x) = x$

$$\Rightarrow P[(Q-P)]x = Px$$

$$\Rightarrow PQx - (P^2)x = Px$$

$$\Rightarrow Px - Px = Px$$

$$\Rightarrow (Q-P)(x) = Qx$$

$$\Rightarrow Px = 0$$

$$\Rightarrow x \in M^{\perp}.$$

Hence $x \in N \cap M^{\perp}$.

So $\quad R \subseteq N \cap M^{\perp}$..................(2)

From (1) and (2) we get $R = N \cap M^{\perp}$

Hence $Q - P$ is the projection on $N \cap M^{\perp}$.

END.