PDE : (M.Sc. Sem-II)
***By* : Dr. Loknath Rai**
Prof. and Head of Mathematics
Patna University, Patna

Email : loknathrai7@gmail.com
Call : 9006632911

COUNTOUR INTEGRATION

In this lession basically we will put up our concentration on Contour Integration and some fundamentals about Path integration Cauchy's integral formula etc.

1.1 Contour Integration :– It is a method of evaluating certain integral along a path in the complex plane.

➲ The countour integration is directly result of residues which is very popular in complex analysis.

➲ The contour integration is very helpful in evulating of integrals along the real line that are not readily found by using the only real variable calculus.

➲ Now, here our aim is to discuss :

Three types of integrations :

(a) Direct integration of a complex – valued function along a curve in complex plane.

(b) Application of the Cauchy integral formula.

(c) Application of the residue theorem.

1.2 Curve in complex plane :– A curve is nothing but of continuous function from a closed interval of the real line to the complex plane.

$$\gamma : [a, b] \to C$$

Note : The curve in complex plane. We also defined it by the technique of parametrization from a closed interval in which parameter is allowed to vary.

1.3 Contour :– A contour is a class of piecewise smooth curves that gives a new curve (Contour) that is if γ_i for γ_i for $1 \le i \le n$ are curves such that end point of γ_i is begining of γ_{i+1} we then define $\gamma_1 + \gamma_2 + ... + \gamma_n$ as a contour and denote it by Γ.

Hence

$$\Gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$$

1.4 Contour Integrals :– In general the contour integral is the sum of integrals over the directed smooth curve (γ_i) that making the contour.

i.e. $$\int_\Gamma f = \int_{\gamma_2} f + \;....... + \int_{\gamma_n} f \qquad .....e_1$$

➲ Integral of comlpex-valued function over an interval $[a, b]$

Let $f:[a, b]\to C$

defined by

$$f(t)=u(t)+i\upsilon(t)$$

where $u(t)=$ Real part of $f(t)$

$\upsilon(t)=$ Imaginary part of $f(t)$

Also $u(t)$ and $\upsilon(t)$ are continuous functions on $[a, b]$

$$\int_a^b f(t)\,dt=\int_a^b u(t)\,dt \;+\; i\int_a^b \upsilon(t)\,dt \qquad \text{........} e_2$$

Moreover

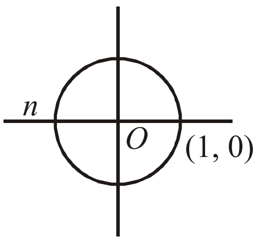
Let $f:c\to c$ be a continuous function on directed smooth curve γ and let z is coming from γ then

$$\int_\gamma f(z)\,dz=\int_a^b f(\gamma(t))\gamma'(t)\,dt \qquad \text{.....}\ e_3$$

The definition given in e_3 is well defined.

Example : Evaluate $\int_{|z|=1}\frac{1}{z}dz$

Solution : Here γ is a unit circle whose parametrisation is as.



$z=\gamma(t)=e^{it};\;\; t\in[0, 2\pi]$

By e_3

$$\int_{\gamma|z|=1}\frac{1}{z}dz=\int_0^{2\pi}\frac{1}{e^{1t}}ie^{it}dt=i\int_0^{2\pi}dt=2\pi i$$

Cauchy-Integral formula :

Let $f:U\to C$ where U is an open subset of complex plane C overwhich f is holomorphic and let γ is a boundary of small enough closed disk oriented in anti-clockwise direction contained in U. Then for every a belonging to interior of close disk.

We have

$$f^n(a)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-a)^{n+1}}dz \qquad \text{(Cauchy's differentiation theorem)} \;\; \text{......}\ (e_4)$$

Special case :

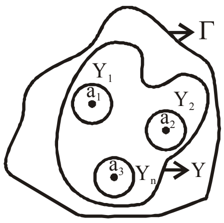
$$f(a)=\frac{1}{2\pi i}\int\frac{f(z)}{(z-a)}dz \text{ for } n=0 \qquad \ldots\ldots \qquad e_5$$

Residue theorem :

Let T be a simply connected open subset of complex plane. C containing a finite sequence of singular points $a_1, a_2, \ldots. a_n$ which are inclosed by the contour γ then integral

$$\int_\gamma f(z)dz=2\pi i\sum_{k=1}^{n} I(\gamma, a_k)\text{Res}(f, a_k)$$

If winding number $I(\gamma, a_k)=1$



then $\int_\gamma f(z)dz=2\pi i\{\text{sum of residues at } a_k; 1\le k\le n\}$

Where γ is positively oriented simple closed curve.

Example : Evaluate $\int_{-\infty}^{\infty}\frac{dx}{x^2+1}$

Solution : The only singular point is at i,

consider; $\frac{1}{z^2+1}=\frac{1}{(z-i)^2(z+1)^2}$

Thus
$$\int_{-a}^{a}\frac{1}{z^2+1}dz=\int\frac{\frac{1}{(z+i)^2}}{(z-i)^2}dz=\int\frac{f(z)}{(z-i)^2}dz-I_1$$

$$I_1=\int_{Arc}\frac{f(z)}{z^2+1}dz \qquad \text{where } f(z)=\frac{1}{(z+i)^2}$$

By using e_4 (Cauchy's differentiation theorem)

$$\int\frac{1}{z^2+1}dz=2\pi i f'(1)=2\pi i\left[\frac{-2}{(z+1)^3}\right]_{z=i}$$

$$= 2\pi i\left[\frac{-2}{-8i}\right] = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = \frac{\pi}{2} - \int_{Arc}\frac{f(z)}{z^2+1}dz = \frac{\pi}{2} \text{ as } \int\frac{f(z)}{z^2+1}dz \longrightarrow 0 \text{ as } a \to \infty$$

Aliter; (By Caucy's residue theorem)

Now, $\int_{-a}^{a}\frac{1}{z^2+1}dz = 2\pi i\{\text{sum of residues}\}$

Thus we have

$$\text{Res}(f, i) = \frac{1}{4i}$$

Hence,

$$\int\frac{1}{z^2+1}dz = 2\pi i\frac{1}{4i} = \frac{\pi}{2} \qquad \text{..... } e_6$$

On comparing real & imaginary part both sides in e_6

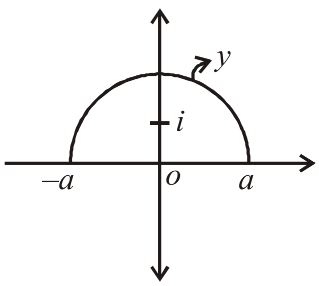
$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{x^2+1}dx = \frac{\pi}{2}$$

Example (2) : Evaluate $\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}dx$

Solution : Let us consider

$$\int_C\frac{e^{iz}}{z^2+1}dz \qquad \left(\because \frac{e^{iz}}{z^2+1} \text{ is entire function excepting at } z : z^2+1=0\right)$$

the only singularity which contain the contour is $z = i$



$$\text{Res }(f, i) = (z-i)\frac{e^{iz}}{(z-i)(z+1)} \text{ as } z \to i$$

$$\text{Res }(f, i) = \frac{e^{-1}}{2i}$$

Now by Residue theorem

$$\int_C \frac{e^{iz}}{z^2+1}dz = \int_{-a}^{a} \frac{e^{iz}}{z^2+1}dz + \int_{Arc=\gamma} \frac{e^{iz}}{z^2+1}dz$$

But $\int_{-a}^{a} \frac{e^{iz}}{z^2+1}dz + \int_{Arc=\gamma} \frac{e^{iz}}{z^2+1}dz = 2\pi i\left(\frac{e^{-1}}{2i}\right) = \pi e^{-1}$

Also by ML Lemma $(\text{As } a \to \infty)$

We have

$$\left|\int_{Arc} \frac{e^{iz}}{z^2+1}dz\right| \le ML \to 0 \text{ (Do yourself)}$$

Where $M \to \max^m \; of \; |f(z)|$ on Arc γ

$L \to$ Length of semicircle $\gamma = \pi a$

Hence $\int_{-a}^{a} \frac{e^{iz}}{z^2+1}dz = \pi e^{-1}$

Putting $y = 0,$ both sides (i.e. comparing real parts)
and taking $a \to \infty$
We get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx = \frac{\pi}{e}$$

Assignment :– Discuss the winding number of pole & integral with winding number greater than 1 of a function.