e-content (lecture-23)

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Theorem and problem on projection.

Theorem: If P is a projection a Hilbert space H ,then

(1) P a positive operator i.e $P \geq O$.

(2) $O \leq P \leq I$

(3) $\| Px\| \leq \|x\| \quad \forall \ x \in H$.

(4) $\| Px\| \leq 1$

Proof: It is given that P is a projection a Hilbert space H .Then $P^2 = P \ \ and \ P^* = P$.

(1) Let x be any vector in H. Then

$$(Px, x) = (PPx, x)$$

$$= (Px, P^*x) = (Px, Px) = \| Px\|^2 \geq 0 \quad \forall \ x \in H.$$

Hence $(Px, x) \geq 0 \quad \forall\ x \in H$. So P a positive operator i.e $P \geq O$.

(2) Since P is a projection a Hilbert space H therefore

$I - P$ is also a projection a Hilbert space H

Hence we have $I - P \geq O$ i.e $P \leq I$. But $O \leq P$.

So $O \leq P \leq I$.

(3) Let x be any vector in H. If M is the range of P.

Then $M^{\perp}$ is the range of $I - P$.

Now $Px \in M$ $and\ I - Px \in M^{\perp}$. Hence $Px\ and\ I - Px$

are orthogonal vectors .So we have

$$\| Px + (I - P)x\|^2 = \| Px\|^2 + \| (I - P)x\|^2$$

$$\Rightarrow \| x\|^2 = \| Px\|^2 + \| (I - P)x\|^2$$

$$\Rightarrow \| x\|^2 \geq \| Px\|^2$$

$$\Rightarrow \| Px\| \leq \|x\| \quad \forall\ x \in H.$$

(4) We have $\|P\| = Sup\{\, \| Px\| : \|x\| \leq 1\}$.But we have $\| Px\| \leq \|x\| \quad \forall\ x \in H$. Hence

$$Sup\{\, \| Px\| : \|x\| \leq 1\} \leq 1 \Rightarrow \| Px\| \leq 1.$$

Problem: Show that an idempotent operator on a Hilbert space H is a projection on H if and only if it is normal.

Solution: Let P is an idempotent operator on a Hilbert space H .So $P^2 = P$.

Let P be a projection on H .Then $P^* = P$.

We have $PP^* = P^*P^* = P^*P$ Hence P is normal.

Conversely Let $PP^* = P^*P$ To prove $P^* = P$

For any vector $x\ in\ H$we have

$(Py, Py) = (y, P^*Py) = (y, PP^*y) = (P^*y, P^*y)$

Let x be any vector in H. Let $y = x - Px$

Then $Py = P(x - Px) = Px - PPx = Px - Px = 0$

$0 = P^*y = P^*(x - Px) = P^*x - P^*Px$

$P^*x = P^*Px \ \forall\ x \in H \Rightarrow P^* = P^*P$

$P = (P^*)^* = (P^*P)^* = P^*P = P^*$.So P is self adjoint and $P^2 = P$.Therefore P is a projection on H.

End