GAME THEORY-III
(M.Sc. Sem-III)

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GRAPHIC SOLUTION OF 2×*N* AND *M*×2 GAMES

The procedure described in the last section will generally be applicable for any game with 2×2 payoff matrix unless it possesses a saddle point. Moreover, the procedure can be extended to any square payoff matrix of any order. But it will not work for the game whose payoff matrix happens to be a rectangular one, say $m \times n$. In such cases a very simple graphical method is available if either m or n is two. The graphic short-cut enables us to reduce the original $2 \times n$ or $m \times 2$ game to a much simpler 2×2 game. Consider the following $2 \times n$ game :

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{pmatrix} B_1 & B_2 & \cdots & B_n \\ A_1 & a_{12} & \cdots & a_{1n} \\ A_2 & a_{22} & \cdots & a_{2n} \end{pmatrix} \end{array}$$

It is assumed that the game does not have a saddle point. Let the optimum mixed strategy for A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_1 + p_2 = 1$. The average (expected) payoff for A when he plays S_A against B's pure moves $B_1, B_2, \ldots., B_n$ is given by

B's pure move	A's expected payoff $E(p)$
B_1	$E_1(p_1) = a_{t1}p_1 + a_{21}p_2 = a_{11}p_1 + a_{21}(1 - p_1)$
B_2	$E_2(p_1) = a_{12}p_1 + a_{22}p_2 = a_{12}p_1 + a_{22}(1 - p_1)$
$\vdots$	$\vdots$
B_n	$E_n(p_1) = a_{1n}p_1 + a_{2n}p_2 = a_{1n}p_1 + a_{2n}(1 - p_1)$

According to the maximum criterion for mixed strategy games, player A should select the values of p_1 and p_2 so as to maximize his minimum expected payoffs. This may be done by plotting the expected payoff lines :

$$E_j(p_1) = (a_{ij} - a_{2j})p_1 + a_{2j} \; (j = 1, 2, \ldots n).$$

The highest point on the lower envelope of these lines will give maximum of the minimum (i.e., maximin) expected payoffs to player A as also the maximum value of p_1.

The two lines* passing through the maximin point identify the two critical moves of B which combined with two of A, yield the 2×2 matrix that can be used to determine the optimum strategies of the two players, for the original game, using the results of the previous section.

The $(m\times2)$ games are also treated in the same way where the upper envelope of the straight lines corresponding to B's expected payoffs will give the maximum expected payoff to player B and the lowest point on this then gives the minimum expected payoff (minimax value) and the optimum value of q_1.

SAMPLE PROBLEMS

1. Solve the following 2×2 game graphically :

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

Solution. Clearly, the problem does not possess a saddle point. Let the player A play the mixed strategy

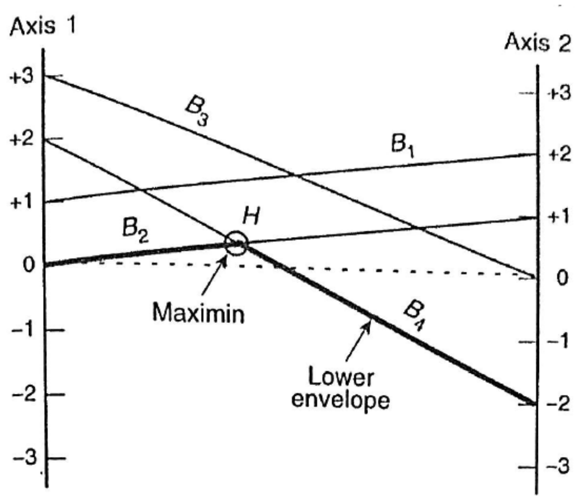
$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given

by

B's pure move	A's expected payoff $E\ (p_1)$
B_1	$E_1(p_1) = p_1 + 1$
B_2	$E_2(p_1) = p_1$
B_3	$E_3(p_1) = -3p_1 + 3$
B_4	$E_4(p_1) = -4p_1 + 2$

These expected payoff equations are then plotted as functions of p_1 as shown in figure. Which shows the payoffs of each column represented as points on two vertical axis 1 and 2, unit distance apart. Thus line B_1 joins the first payoff element 2 in the first column represented by +2 on axis 2 and the second payoff element I in the first column represented by +1 on axis 1. Similarly, lines B_2 B_3 and B_4 join the corresponding representation of payoff elements in the second, third and fourth columns. Since the player A wishes to maximize his minimum expected payoff we consider the highest point of intersection H on the lower envelope of the A's expected payoff equations. This point H represents the maximum expected value of the game for A. The lines B_2 and B_4, passing through H, define the two relevant moves B_2 and B_4 that alone B needs to play. The solution to the original 2+4 game, therefore, boils down that of the simpler game with the 2×2 payoff matrix :

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \end{array}$$



The maximin value

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_2 & B_4 \\ q_2 & q_4 \end{bmatrix}$

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2-(-2)} = 2/5,\ p_2 = 1-p_1 = 3/5,$$

$$\text{d}\, q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5,\ q_4 = 1-q_1 = 1/5.$$

Hence, the solution to the game is

(i) the optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$

(ii) the optimum strategy for B is : $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$

and (iii) the expected value of the game is $v = \dfrac{2\times 1 - 0\times(-2)}{1+2-(0-2)} = \dfrac{2}{5}$.

2. Obtain the optimal strategies for both-persons and the value of the game for zero-sum two person game whose payoff matrix is as follows :

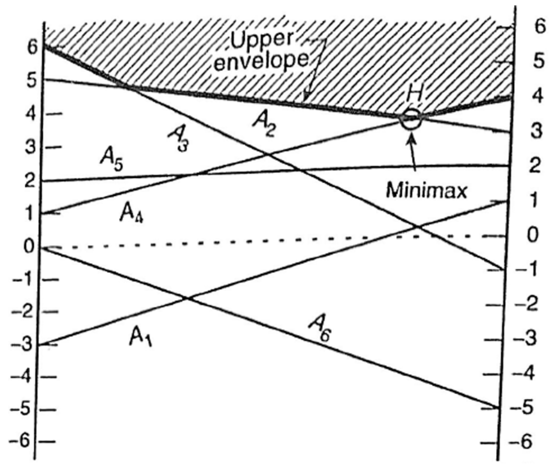
$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}$$

Sol. Clearly, the given problem does not possess any saddle point. So, let the player B play the mixed strategy

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then B's expected payoffs against A's pure moves are given by

A's pure move	B's expected payoff $E(q_1)$
A_1	$E_1(q_1) = 4q_1 - 3$
A_2	$E_2(q_1) = -2q_1 + 5$
A_3	$E_3(q_1) = -7q_1 + 6$
A_4	$E_4(q_1) = 3q_1 + 1$
A_5	$E_5(q_1) = 2$
A_6	$E_6(q_1) = -5q_1$

The expected payoff equations are then plotted as functions of q_1 as shown in figure.



The minimax value

Since, the player B wishes to minimize his maximum expected payoff, we consider the loose point of intersection H on the upper envelope of B's expected payoff equations. This point B represents the minimax expected value of the game for player B. The lines A_2 and A_4 passing through H, define the two relevant moves A_2 and A_4 that alone the player A needs to play. The solution to the original 6×2 game, therefore, reduces to that of the simpler game with 2×2 payoff matrix :

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

If we now, let

$$S_A = \begin{bmatrix} A_2 & A_4 \\ p_1 & p_2 \end{bmatrix},\ p_1 + p_2 = 1;\ S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix},\ q_1 + q_2 = 1$$

then using the usual method of solution for 2×2 games, the optimum strategies can easily be obtained as

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix},\ S_B = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$$

and the value of the game as v = 17/5.