e-content (lecture-22)

by

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Theorem on projection .

Theorem: If P is a projection on a Hilbert space H with range M and null space N ,then $M \perp N \Leftrightarrow P$ is self-adjoint and in this case $N = M^{\perp}$.

Proof: Suppose P is a projection on a Hilbert space H with range M and null space N.Then $H = M \oplus N$.

Suppose that $M \perp N$ $\quad To\ prove$ P is self- adjoint.

Let $z \in H$ Then z can be uniquely written as

$z = x + y$ where $x \in M$ $and\ y \in N$.

Now we have $\quad (Pz, z) = (x, z)$

$$= (x, x + y)$$

$= (x, x) + (x, y)$

$= (x, x)$

[since $x \in M$ and $y \in N$ and $M \perp N$ so $(x, y) = 0$]

Also $(P^* z, z) = (z, Pz) = (z, x)$

$= (x + y, x)$

$= (x, x) + (x, y)$

$= (x, x)$

Hence $\forall z \in H$

we have $(Pz, z) = (P^* z, z)$

$\Rightarrow P = P^* \Rightarrow$ P is self- adjoint.

Conversely Suppose that P is self- adjoint.

To prove $M \perp N$.

Let $x \in M$ and $y \in N$

Then $(x, y) = (Px, y)$

$= (x, P^* y)$

$= (x, Py)$

[Since $y \in N \Rightarrow Py = 0$]

$$= (x, 0) = 0.$$

Hence $\forall x \in M \ and \ \forall y \in N, (x, y) = 0 \Rightarrow M \perp N$.

Second Part: To prove $N = M^{\perp}$.

Let $x \in N$ Then $M \perp N \Rightarrow x \perp M$

$$\Rightarrow x \in M^{\perp}$$

$$\Rightarrow N \subseteq M^{\perp} \text{.....(1)}$$

Let N be proper subset of $M^{\perp}$. Then N is a proper closed linear subspace of the Hilbert space $M^{\perp}$.

So there exists a non zero vector z_0 in $M^{\perp}$ such that

$z_0 \perp N$ .But $z_0 \in M^{\perp} \Rightarrow z_0 \perp M$

Since $z_0 \perp N \ and z_0 \perp M \ and \ H = M \oplus N$.

Hence $z_0 \perp H \Rightarrow z_0 = 0$ we get a contradiction.

So we must have $N = M^{\perp}$.

END.