Differential and integral Equation

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1 Equicontinuous Function

1.1 Equicontinuous Set of Function

Let a set of functions F={f : defined on a rea interval I } is said to be equicontinuous if for a given $\epsilon > 0$, there exist a $\delta_\epsilon > 0$ independent of f and also t, $\bar{t} \in I$ such that

$$\left| f(t) - f(\bar{t}) \right| < \epsilon \quad \textbf{whenever} \quad \left| t - \bar{t} \right| < \delta_\epsilon$$

Ascoli-Arzella Theorem: On a bounded real interval I, let

$$F = \{f : f \ \text{ be defined on } \ I\}$$

be infinite, uniformly bounded equicontinuous set of function f. Then F contains a sequence $\{f_n\}$ which is uniformly convergent on I.

Proof :- Let a sequence $\{r_k\}$, $k = 1, 2, 3, ...$ be a rational on I, enumerate in same. order. Since F is uniformly bounded over I. Hence there exist a number $M > 0$ such that

$$f(x) \leq M \ \ \forall \ \ x \in I, \ \ \forall \ \ f \in F$$

the set of number $\{f(r_1)\}$ is bounded on I.

So there exists a sequence of distinct real valued function $\{f_{n_1}\}$ such that the sequence $\{f_{n_1}(r_1\}$ is convergent on I

Similarly, the set of numbers $\{f_{n_1}(r_2\}$ is also bounded and and convergent consequently the sequence of numbers $\{f_{n_2}(r_2\}$ is convergent on I. Continuing in this way we get a sequence of real valued function $\{f_{n_k}\}$ $n, k = 1, 2, 3, ...$, which is convergent on each rational $\{r_k\}$ $k = 1, 2, 3, ...$ i.e.,

$$\{f_{n_k}\} \ \text{ is pointwise convergent on } \ I$$

Now, $\{f_n\}$ is pointwise convergent on I. We need to prove that $\{f_n\}$ is uniformlly convergent on I. For a given $\epsilon > 0$ and for a rational $\overline{r_k}$ in I, $\exists$ a positive integer $N_\epsilon(\overline{r_k})$ such that

$$\left| f_n(\overline{r_k}) - f_m(\overline{r_k}) \right| < \epsilon \quad \forall \ \ n, m \geq N_\epsilon(\overline{r_k})$$

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For this some $\epsilon > 0$, $\exists$ a $\delta_\epsilon > 0$ such that $\forall\ t, \bar{t} \in I$ and $\forall\ f \in F$, $f_n \in F$

$$\left|f(t) - f(\bar{t})\right| < \epsilon \text{ whenever } \left|t - \bar{t}\right| < \delta_\epsilon$$

Hence δ_ϵ is independent of t, $\bar{t} \in I$ and $f \subset F$
Since $f_n \subset F \quad \forall\ n \in \mathbb{N}$. So we have

$$\left|f_n(t) - f_n(\overline{r_k})\right| < \epsilon \text{ whenever } \left|t - \overline{r_k}\right| < \delta_\epsilon$$

Now, divided the whole interval I into finite number of subinterval $I_1, I_2, I_3, ..., I_n$ such that the length of the largest subinterval is less than δ_ϵ. If we select any rational $t \in I$, it will be in some subinterval $I_1, I_2, I_3, ..., I_n$. Let $\overleftarrow{r_p} \in I$ again it will be in subinterval I_k. Then

$$\begin{aligned}\left|f_n(t) - f_m(t)\right| &= \left|f_n(t) - f_n(\overline{r_k}) + f_n(\overline{r_k}) - f_m(\overline{r_k}) + f_m(\overline{r_k}) - f_m(t)\right| \\ &\leq \left|f_n(t) - f_n(\overline{r_k}\right| + \left|f_n(\overline{r_k}) - f_m(\overline{r_k})\right| + \left|f_m(\overline{r_k}) - f_m(t)\right| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon\end{aligned} \tag{1}$$

so assume

$$N_\epsilon = max\Big\{N_\epsilon(\overline{r_1}), N_\epsilon(\overline{r_2}), N_\epsilon(\overline{r_3}), ..., N_\epsilon(\overline{r_k})\Big\}$$

then we have

$$\left|f_n(t) - f_m(t)\right| < \epsilon \quad \forall\ n, m \geq N_\epsilon \tag{2}$$

$\implies \{f_n\}$ is uniformly convergent on I

1.2 Gronwall's inequality

Statement: If $u(t)$ and $v(t)$ be non-negative and continuous on $[a, b]$ and "C" be a non-negative constant. Again if

$$v(t) \leq Ct \int_a^t u(s)v(s)ds, \quad t \in [a, b]$$

Then

$$v(t) \leq C \exp\Big(\int_a^t u(s)ds\Big), \quad t \in [a, b]$$

and if $c = 0$

$$\implies \quad v(t) = 0 \quad \forall\ t \in [a, b]$$

Proof :- Given C be non-negative $\implies C \geq 0$.

Case: I If $C > 0$, Suppose

$$V(t) = C + \int_a^t u(s)v(s)ds, \quad t \in [a, b] \tag{3}$$

Then,

$$v(t) \leq Ct \int_a^t u(s)v(s)ds$$

Then,

$$v(t) \leq V(t) \geq C \geq 0$$

$$\implies V(t) > 0 \qquad \forall \;\; t \in [a, b]$$

Now,

$$\begin{aligned} V(t) = C + \int\limits_a^t u(s)v(s)ds, \qquad a \leq t \leq b \\ \implies \quad V'(t) = v(t)u(t) \leq u(t)V(t) \\ \implies \quad \tfrac{V'(t)}{V(t)} \leq u(t) \end{aligned} \tag{4}$$

Integrating both side of equation (4), then

$$\begin{aligned} \int \tfrac{V'(t)}{V(t)} dt \leq \int u(t)dt \\ \implies \quad \int\limits_a^t \tfrac{V'(s)}{V(s)} ds \leq \int\limits_a^t u(s)ds \\ \implies \quad \Big[\log V(s)\Big]_a^t \leq \int\limits_a^t u(s)ds \\ \implies \quad \log V(t) - \log V(a) \leq \int\limits_a^t u(s)ds \end{aligned} \tag{5}$$

Comparing equations (3) and (5) , we get

$$\begin{aligned} C = V(a) \\ \implies \quad \log \tfrac{V(t)}{V(a)} \leq \int\limits_a^t u(s)ds \\ \implies \quad \tfrac{V(t)}{C} \leq \exp\Big(\int\limits_a^t u(s)ds\Big) \end{aligned} \tag{6}$$

$$\therefore \quad V(t) \leq C \exp\Big(\int_a^t u(s)ds\Big) \tag{7}$$

$$\therefore \quad v(t) \leq V(t) \leq C \exp\Big(\int_a^t u(s)ds\Big) \tag{8}$$

Case: II If $C = 0$ then equation (7)

$$\implies \quad 0 \leq V(t) \leq 0, \qquad \therefore \quad V(t) = 0 \tag{9}$$

Hence from equations (8) and statement, we get

$$v(t) = 0. \quad \forall \quad t \in [a, b]$$

...............All the best.........