e-content (lecture-21)

by

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Theorem and problems on Normal and Unitary operators .

Theorem: If T is an operator on a Hilbert space H ,

then T is unitary $\Leftrightarrow$ it is an isometry isomorphism of H onto itself.

Proof : Since T is an unitary operator on a Hilbert space H so T is invertible and hence it is onto .

Also $T^*T = I$ so we have $\|Tx\| = \|x\| \;\; \forall x \in H$.

Hence T is an isometry isomorphism of H onto itself.

Conversely suppose that T is an isometry isomorphism of H onto itself.

To prove T is unitary.

Since T is an isometry isomorphism of H onto itself hence T is one-one and onto so T^{-1} exists

Also T is an isometry isomorphism

$$\Rightarrow \|Tx\| = \|x\| \quad \forall x \in H.$$

$$\Rightarrow T^*T = I$$

$$\Rightarrow T^*(TT^{-1}) = T^{-1}$$

$$\Rightarrow T^*I = T^{-1}$$

$$\Rightarrow T^* = T^{-1}.$$

Now T is invertible

so $T^{-1}T = I = TT^{-1}$

$\Rightarrow T^*T = I = TT^*$ [since $T^* = T^{-1}$]

Hence T is unitary.

Problem: If T is a normal operator on a Hilbert space H and α is any scalar then $T - \alpha I$ is also normal .

Solution: Since T is a normal

therefore $T^*T = TT^*$.

Now $(T - \alpha I)\,(T - \alpha I)^* = (T - \alpha I)(T^* - \overline{\alpha} I^*)$

$$= (T - \alpha I)(T^* - \overline{\alpha} I)$$

$$(T - \alpha I)\,(T - \alpha I)^* = TT^* - \overline{\alpha}T - \alpha T^* + |\alpha|^2 I \quad ....(1)$$

Also $(T - \alpha I)^*(T - \alpha I) = (T^* - \overline{\alpha} I^*)\,(T - \alpha I)$

$$= T^*T - \overline{\alpha}T - \alpha T^* + |\alpha|^2 I \quad ....(2)$$

Since $T^*T = TT^*$ so from (1) and (2) we get

$$(T - \alpha I)\,(T - \alpha I)^* = (T - \alpha I)^*(T - \alpha I).$$

Problem: If T is an arbitrary operator on a Hilbert space H and α and β are any scalars such that

$|\alpha| = |\beta|$ then $\alpha T + \beta T^*$ is also normal .

Solution: We have

$$(\alpha T + \beta T^*)^* = (\alpha T)^* + (\beta T^*)^*$$

$$= \overline{\alpha} T^* + \overline{\beta} T^{**}$$

$$= \overline{\alpha} T^* + \overline{\beta} T \quad .....(1)$$

Now $(\alpha T + \beta T^*)(\alpha T + \beta T^*)^*$

$$= (\alpha T + \beta T^*)(\overline{\alpha} T^* + \overline{\beta} T) \quad \text{[from (1)]}$$

$$= |\alpha|^2 TT^* + \alpha\overline{\beta} T^2 + \beta\overline{\alpha} T^* T^* + |\beta|^2 T^* T \quad ...(2)$$

Again $(\alpha T + \beta T^*)^*(\alpha T + \beta T^*)$

$$= (\overline{\alpha} T^* + \overline{\beta} T)(\alpha T + \beta T^*)$$

$$= |\alpha|^2 T^* T + \alpha \overline{\beta} T^2 + \beta \overline{\alpha} T^* T^* + |\beta|^2 T T^* \quad \text{...(3)}$$

Since $|\alpha| = |\beta|$ hence from (2) and (3) we get

$$(\alpha T + \beta T^*)(\alpha T + \beta T^*)^* = (\alpha T + \beta T^*)^*(\alpha T + \beta T^*)$$

Therefore $\alpha T + \beta T^*$ is also normal .

END