M.S c Mathematics –SEM 3 Functional Analysis-L-9 CC-11 Unit 2

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Theorem based on Inner Product Space

If L is an inner product space show that $\sqrt{(x,x)}$ has the properties of a norm.

Solution: Let L be an inner product space and let us write

$||x|| = \sqrt{(x,x)}$. This relation will satisfy all the properties of a norm if we show that

(i) $||x|| \geq 0$ and $||x|| = 0 \Rightarrow x = 0$

(ii) $||\alpha x|| = |\alpha|\,||x||$

(iii) $||x+y|| \leq ||x|| + ||y||$

We shall prove these one by one

(i) **By our definition of an inner product space we have**

$$(x,x) \geq 0 \text{ or } ||x||^2 = (x,x)$$

Therefore $||x|| \geq 0$ *and* $||x|| = 0 \Leftrightarrow x = 0$

(ii) **We have** $||\boldsymbol{\alpha x}||^2 = (\boldsymbol{\alpha x}, \boldsymbol{\alpha x})$

$$= \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{\alpha x})$$

$$= \boldsymbol{\alpha\bar{\alpha}}\ (\boldsymbol{x}, \boldsymbol{x}) = |\boldsymbol{\alpha}|^2\ ||\boldsymbol{x}||^2$$

$\boldsymbol{Taking}$ **square root of both sides ,we get**

$$||\boldsymbol{\alpha x}|| = |\boldsymbol{\alpha}|||\boldsymbol{x}||$$

$(\boldsymbol{iii})$ **We have**

$$||\boldsymbol{x}+\boldsymbol{y}||^2 = (\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y})$$

$$= (\boldsymbol{x}, \boldsymbol{x}) + (\boldsymbol{x}, \boldsymbol{y}) + (\boldsymbol{y}, \boldsymbol{x}) + (\boldsymbol{y}, \boldsymbol{y})$$

$= ||x+y||^2 = ||x||^2 + (x,y) + \overline{(x,y)} + ||y||^2$

$= ||x||^2 + 2Re(x,y) + ||y||^2$

$\leq ||x||^2 + 2|(x,y)| + ||y||^2$

$\leq ||x||^2 + 2\sqrt{(x,x)}.\sqrt{(y,y)} + ||y||^2$

$= ||x||^2 + 2||x||.||y|| + ||y||^2$

$= (||x|| + ||y||)^2$

Thus $||x+y||^2 \leq (||x|| + ||y||)^2$

Taking square root of both sides ,we get

$$||x+y|| \leq ||x|| + ||y||$$