GAME THEORY-II
(M.Sc. Sem-III)

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GAMES WITHOUT SADDLE POINTS – MIXED STRATEGIES

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position – the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategy to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probablistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A are expect to win and the least which player *B* can lose. The expected payoff to a player in a game to arbitraty payoff matrix (a_{ij}) of order m×n is defined as :

$$E(p, q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j = p^T Aq$$

where p and q denote the mixed strategies for players A and B respectively.

Maxima-Minimax Criterion. Consider an *m*×*n* game (a_{ij}) without any saddle point, its stragteies are mixed. Let $p_1, p_2, ..., p_m$ be the probabilities with which player *A* will play his move $A_1, A_2, ..., A_m$ respectively; and let $q_1, q_2, ..., q_n$ be the probabilities with which player *B* will pay his moves $B_1, B_2, ..., B_n$ respectively. Obviously, $p_i \geq 0 \ (i = 1, 2, ..., m), \ q_j \geq 0 (j = 1, 2, ..., 0)$ and $p_1 + p_2 + ... + p_m = 1; \ q_1 + q_2 + ... + q_n = 1.$

The expected payoff function for player *A*, therefore, will be given by

$$E(p, q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} a_j$$

Making use of maximin-minimax criterion, we have

For Player *A*.

$$\underline{v} = \max_p . \min_q . \ E(p, q) = \max_p . \left[\max_j . \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\} \right]$$

$$= \max_p . \left[\min_j . \left\{ \sum_{i=1}^{m} p_i a_{i1}, \ \sum_{i=1}^{m} p_i a_{i1}, \ ...., \ \sum_{i=1}^{m} p_i a_{in} \right\} \right]$$

Here, $\min . \left\{ \sum_{i=1}^{m} p_i a_{ij} \right\}$ denotes the expected gain to player *A*, when player *B* uses his *j*th pure string.

For player B.

$$\underline{v} = \min_{q} . \left[\max_{i} . \left\{ \sum_{j=1}^{n} a_j a_{1j}, \sum_{j=1}^{n} a_j a_{2j}, ..., \sum_{j=1}^{n} a_j a_{mj} \right\} \right]$$

Here $\max . \left\{ \sum_{j=1}^{n} a_j a_{ij} \right\}$ denotes the expected loss to player B when player A uses his ith strategy.

The relationship $\underline{v} \le \bar{v}$ holds good in general and when p_i and q_j correspond to the opposite strategies the relation holds in 'equality' sense and the expected value for both the players because euqal to the optimum expected value of the game.

Definition : A pair of strategies (p, q) for which $\underline{v} = \bar{v} = v$ is called a saddle point of E (p, q).

Theorem : For any 2×2 two-person zero-sum game without any saddle point have payoff matrix for player A.

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

the optimum mixed strategies

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$. The value v of the game to A is given by

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Proof : Let a mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1 + p_2 = 1$. Thus, if player B moves B_1 the net expected gain of A will be

$$E_1(p) = a_{11}p_1 + a_{21}p_2$$

and if B moves B_2, the net expected gain of A will be

$$E_2(p) = a_{12}p_1 + a_{22}p_2$$

Similarly, if B plays his mixed strategy $S_n = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$, where $q_1 + q_2 = 1$, then B's net expected loss will be

if A plays A_1, and $\quad E_1(q) = a_{11}q_1 + a_{12}q_2$

if A plays A_2, $\quad E_2(q) = a_{21}q_1 + a_{22}q_2$

The expected gain of player A, when B mixes his moves with probabilities q_1 and q_2 is, therefore, given by

$$E(p, q) = q_1[a_{11}p_1 + a_{21}p_2] + q_2[a_{12}p_1 + a_{22}p_2]$$

Player A would always try to mix his moves with such probabilities so as to maximize his expected gain.

Now,
$$E(p, q) = q_1[a_{11}p_1 + a_{21}(1 - p_1)] + (1 - q_1)[a_{12}p_1 + a_{22}(1 - p_1)]$$

$$= [a_{11} + a_{22} - (a_{12} + a_{21})]p_1q_1 + (a_{12} - a_{22})p_1 + (a_{21} - a_{22})q_1 + a_{22}$$

$$= \lambda\left(p_1 - \frac{a_{22} - a_{21}}{\lambda}\right)\left(q_1 - \frac{a_{22} - a_{12}}{\lambda}\right) + \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$$

where $\lambda = a_{11} + a_{22} - (a_{12} + a_{21})$.

We see that if A chooses $p_1 = \frac{a_{22} - a_{21}}{\lambda}$, he ensures an expected gain of at least $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. Similarly, if B chooses $q_1 = \frac{a_{22} - a_{12}}{\lambda}$, then B will limit his expected loss to at most $(a_{11}a_{22} - a_{12}a_{21})/\lambda$. These choices of p_1 and q_1 will thus be optimal to the two players.

Thus, we get

$$p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \text{ and } p_2 = 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

$$q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \text{ and } q_2 = 1 - q_1 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})};$$

and
$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Hence, we have

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}},\ \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}; \text{ and } v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Note : The above formula for p_1, p_2, q_1, q_2 and v are valid only for 2×2 games without saddle points.

SAMPLE PROBLEMS

1. For the game with the following payoff matrix, determine the optimum strategies and the value of the game :

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution : Clearly, the given matrix is without a saddle point. So, the mixed strategies of P_1 and P_2 are:

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ p_1 & p_2 \end{bmatrix}, S_{P_2} = \begin{bmatrix} 1 & 2 \\ q_1 & q_2 \end{bmatrix}; p_1 + p_2 = 1 \text{ and } q_1 + q_2 = 1$$

If $E(p, q)$ denotes the expected payoff function, then

$$E(p, q) = 5p_1q_1 + 3(1-p_1)q_1 + p_1(1-q_1) + 4(1-p_1)(1-q_1)$$

$$= 5p_1q_1 - 3p_1 - q_1 + 4 = 5(p_1 - 1/5)(q_1 - 3/5) + 17/5$$

If P_1 chooses $p_1 = 1/5$, he ensures that his expectation is at least 17/5. He cannot be sure of more than 17/5, because by choosing $q_1 = 3/5$, P_2 can keep $E(p_1, q_1)$ down to 17/5. So P_1 might as well settle for 17/5 and P_2 reconcile to 17/5. Hence, the optimum strategies for P_1 and P_2 are

$$S_{P_1} = \begin{bmatrix} 1 & 2 \\ 1/5 & 4/5 \end{bmatrix}, S_{P_2} = \begin{bmatrix} 1 & 2 \\ 3/5 & 2/5 \end{bmatrix}$$

and the value of the game is v = 17/5

2. Consider a “modified” form of “matching biased coins” game problem. The making player is paid Rs. 8.00 if the two coins turn both heads and Re. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being of matching or non-matching player, which one would you choose and what would be your strategy.

Solution : The payoff matrix for the matching player is given by

$$\begin{array}{ccc} & & \textit{Non-matching Player} \\ & & \begin{array}{cc} H & T \end{array} \\ \text{Matching Player} & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 8 & -3 \\ -3 & 1 \end{bmatrix} \end{array}$$

Clearly, the payoff matrix does not possess any saddle point. The players will use matrix strategies. The optimum mixed strategy for matching player is determined by

$$p_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15}, \ p_2 = \frac{11}{15}$$

and for the non-matching player, by

$$q_1 = \frac{1-(-3)}{8+1-(-3-3)} = \frac{4}{15},\ q_2 = \frac{11}{15}$$

The expected value of the game (corresponding to the above strategies) is given by

$$v = \frac{8-3(-3)(-3)}{8+1-1(-3-3)} = -\frac{1}{15}.$$

Thus, the optimum mixed strategies for matching player and non-matching player are given by

$$S_{match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix} \text{ and } S_{non-match} = \begin{bmatrix} H & T \\ 4/15 & 11/15 \end{bmatrix}$$

Clearly, we would like to be the non-matching player.

PROBLEMS

3. Solve the following game and determine the value of the game :

(a) $\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$ (b) $\begin{array}{cc} & Y \\ X & \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$