E - content – Pro(Dr) L N RAI

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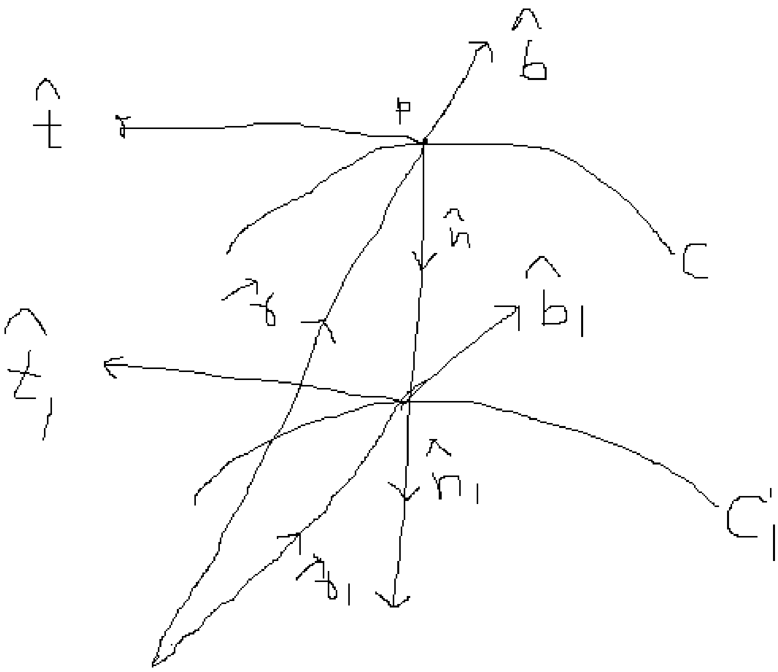
Topic- Differential Geometry

(a) Define Bertrand Curves and show that the distance between corresponding points of two curves is constant

Solution: Two curves C_0 and C_1 having their principal normals in common are called Bertrand curves or conjugate curve



The distance between corespondingpoints of two curves is constant.

Proof

We take their principle normals in the same sense, so that $\hat{n}_1 = \hat{n}$ ....(1)

Let $\vec{r}$ be the position vector of a point on a curve C, the position vector $\vec{r}_1$ of a corresponding point P_1 on the associate Bertrand curve C_1 and C is given by

$\vec{r}_1 = \vec{r} + \lambda\hat{n}$.......................(2)

where λ is a quantity which is function of 'S' and denotes the distance between two corresponding points of two curves

Differentiating (2) w.r.t 's' ,we get

$$\frac{d\vec{r}_1}{ds_1}.\frac{ds_1}{ds} = \hat{t} + \lambda'\hat{n} + \lambda(\zeta\hat{b} - k\hat{t})$$

$$\hat{t}_1\frac{ds_1}{ds} = (1 - \lambda k)\hat{t} + \lambda'\hat{n} + \lambda\zeta\hat{b} \text{.......................}(3)$$

Taking dot product with (1) and (3) ,we get

λ'=0

⇒λ= constant

(b) **The targets at the corresponding points of the two curves are inclined at a constant angle.**

We have

$$\frac{d(t.t_1)}{ds} = \frac{dt}{ds}.t_1 + t.\frac{dt_1}{ds_1}\frac{ds_1}{ds}$$

$$= kn.t_1 + t.k_1.n_1\frac{ds_1}{ds}$$

$$= kn.t_1 + k_1.\frac{ds_1}{ds}\text{t.n} \ \ [\text{since } n_1 = n]$$

$$= 0 \ \ [\text{since } n_1.t_1 = t.n = 0]$$

Integrating t.t_1= constant

Thus if α is the angle between the tangents ,then $cos\alpha = t.t_1$= constant.

Since the principal normals coincide ,it follows that the binormals of the two curves are also inclined at the same constant angle.

(C) Curvature and torsion of either are connected by a linear relation

Since $\lambda' = 0$,the equation (3) becomes

$\hat{t}_1 \frac{ds_1}{ds} = (1 - \lambda k)\hat{t} + \lambda\zeta\hat{b}$....................(4)

Taking the dot product of both sides of (4) with b_1 ,we obtain

$0 = (1 - \lambda k)t.b_1 + \lambda\zeta b.b_1$

Now t.$b_1 = cos\,(90 - \alpha) = sin\,\alpha$

And b.$b_1 = cos\,\alpha$

$$(1 - \lambda k)sin\alpha + \lambda\zeta cos\alpha = 0$$

The above equation shows that there exists a linear reation with constant coefficients between the curvature and torsion of C,.

This relation may be put in the form

$\zeta = (k - \frac{1}{\lambda})tan\alpha$...........................(5)

Now the relation between the curves C and $C_1$is a reciprocal one. The point r is at distance –λ along the principal normal at r_1 and t is inclined at an angle $-\alpha$ with t_1.Hence corresponding to (5), we shall have $\zeta_1 = (k_1 + \frac{1}{\lambda})tan\alpha$ ....(6)

(d) The torsions of the two associate Bertrand cueves have the same sign, and their product is constant.

We know that

$t_1 = tcos\alpha - bsin\alpha$ ....(7)

Comparing (4) of (C) and (7) of (D), we have

$$\frac{ds_1}{ds} = \frac{1-\lambda_k}{cos\alpha} = \frac{\lambda_r}{-sin\alpha}$$

$$cos\alpha = (1-\lambda_k)\frac{ds}{ds_1} \text{............... (8)}$$

$$sin\alpha = -\lambda_r\frac{ds}{ds_1} \text{..................(9)}$$

For the curve C_1, the relation corresponding to (8) and (9) are obtained by replacing λ by –λ by α by $-\alpha$ and interchanging s and s_1.

Thus $cos\alpha = (1+\lambda_{k_1})\frac{ds_1}{ds}$**........................(10)**

And $sin\alpha = -\lambda_{r_1}\frac{ds_1}{ds}$**........................(11)**

Multiplying (8) by (10) and (9) by (11) ,we obtain

$$(1-\lambda_k)(1+\lambda_{k_1}) = cos^2\alpha \text{..........(12)}$$

$$kk_1 = \frac{1}{\lambda^2}sin^2\alpha \text{..................................(13)}$$

The relation (13) shows that the torsions of the two curves have the same sign and their product is constant.