# The Theorem of Blasius(15) 

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## 1 Source and sinks in two-dimension

In two dimensions a source of strength $m$ is such that the flow across any small curve surrounding is $2 \pi m$. Sink is regarded as a source of strength $-m$.

Consider a Circle of radius $r$ with source at its centre. Then radial velocity $q_{r}$ is given by

$$
\begin{equation*}
q_{r}=-\frac{1}{r} \frac{\partial \psi}{\partial \theta} \tag{1}
\end{equation*}
$$

or

$$
\begin{equation*}
q_{r}=-\frac{\partial \phi}{\partial r} \quad \text { as } \quad \frac{\partial \phi}{\partial \theta}=\frac{1}{r} \frac{\partial \psi}{\partial \theta} \tag{2}
\end{equation*}
$$

Then the flow across the circle is $2 \pi r q_{r}$. Hence we have

$$
\begin{equation*}
2 \pi r q_{r}=2 \pi m \quad \text { or } \quad r q_{r}=m \tag{3}
\end{equation*}
$$

or

$$
r\left(-\frac{1}{r} \frac{\partial \psi}{\partial \theta}\right)=m, \quad \text { by }(1) .
$$

Integrating and omitting constant of integration, we get

$$
\begin{equation*}
\psi=-m \theta \tag{4}
\end{equation*}
$$

Using (2) and (3), we Obtain as before,

$$
\begin{equation*}
\phi=-m \log r \tag{5}
\end{equation*}
$$

Equation (4) shows that the streamlines ate $\theta=$ constant, i.e., straight lines radiating from the source. Again (5) shows that the curves of equi-velocity potential are $r=$ constant, i.e., concentric circle with centre at the source.

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## 2 Complex potential due to a source

Let there be a source of strength $m$ at origin then

$$
w=\phi+\iota \psi=-m \log r-\iota m \theta=-m\left(\log r+\iota \log e^{\iota \theta}\right)=-m \log \left(r e^{\iota \theta}\right)=-m \log z
$$

If, however, the source is at $z^{\prime}$, then the complex potential is given by $w=-m \log \left(z-z^{\prime}\right)$. The relation between $w$ and $z$ for source of strengths $m_{1}, m_{2}, \ldots$ situated at the points $z=z_{1}, z_{2}, z_{3} \ldots$ is given by

$$
\begin{array}{r}
w=-m_{1} \log \left(z-z_{1}\right)-m_{2} \log \left(z-z_{2}\right)-m_{3}\left(z-z_{3}\right)-\ldots \\
\text { leading to } \quad \phi=-m_{1} \log r_{1}-m_{2} \log r_{2}-m_{3} \log r_{3}-\ldots \\
\quad \text { and } \quad \psi=-m_{1} \theta_{1}-m_{2} \theta_{2}-m_{3} \theta_{3} \ldots \\
\text { where } \quad r_{n}=\left|z-z_{n}\right| \quad \text { and } \quad \theta_{n}=\arg \left(z-z_{n}\right), \quad n=1,2,3, \ldots
\end{array}
$$

## 3 The Theorem of Blasius.

In a steady two-dimensional irrational motion of and in-compressible fluid under no external forced given by the complex potential $w=f(z)$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force $(X, Y)$ and a couple of moment $M$ about the origin of co-ordinates, then

$$
X-\iota Y=\frac{1}{2} \iota \rho \int_{c}\left(\frac{d w}{d z}\right)^{2} d z, \quad \mathrm{M}=\text { real part of } \quad\left\{-\frac{1}{2} \iota \rho \int_{c} z\left(\frac{d w}{d z}\right)^{2} d z\right\}
$$

Where $\rho$ is the fluid density and integrals are taken round the contour $C$ of the cylinder.
Proof. In the figure of the cylinder in plane $X O Y$. Let $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two neighbouring points on $C$ such that arc $P Q=\delta s$. If $\theta$ be the angle which the tangent $P T$ at $P$ on the contour $C$ makes with $x$ - axis, then

$$
\begin{equation*}
\cos \theta=d x / d s, \quad \sin \theta=d y / d s \tag{6}
\end{equation*}
$$

and the normal at $P$ makes an angle $(\theta+\pi / 2)$ with the $x$-axis. Now, if $p$ denotes the pressure at $p$, the force on unit length of the section $\delta s$ is $p \delta s$ to $C$. Then using (6), we have

$$
\begin{gather*}
X=\int_{c} p \cos (\theta+\pi / 2) d s=-\int_{c} p \sin \theta d s=-\int_{c} p d y, \quad \text { using }  \tag{7}\\
Y=\int_{c} p \sin (\theta+\pi / 2) d s=\int_{c} p \cos \theta d s=\int_{c} p d x, \quad \text { using }  \tag{8}\\
M=\int_{c}[x \dot{p} \sin (\theta+\pi / 2) d s-y \cos (\theta+\pi / 2) d s]=\int_{c} p(x \cos \theta d s+y \sin \theta d s)
\end{gather*}
$$

or

$$
\begin{equation*}
M=\int_{c} p(x d x+y d y), \quad \text { using }(6) \tag{9}
\end{equation*}
$$

Now Bernoulli's equation in this context is

$$
\begin{equation*}
\frac{1}{2} q^{2}+\frac{p}{\rho}=B \quad \text { so that } \quad p=\rho B-\frac{1}{2} \rho q^{2} \tag{10}
\end{equation*}
$$

where $q$ is the fluid velocity, $\rho$ density. Since $\rho$ is constant for an imcompressible fluid, take $\rho B=A$ (a constant). Again $q^{2}=u^{2}+v^{2}$ where $u$ and $v$ ate the velocity components. Then(10) reduces to

$$
\begin{equation*}
p=A=(\rho / 2) \times\left(u^{2}+v^{2}\right) \tag{11}
\end{equation*}
$$

Also,

$$
\begin{equation*}
d w / d z=-u+\iota v \quad \text { or } \quad-d w / d z=u-\iota v \tag{12}
\end{equation*}
$$

using (11),(8),(7) and (9) reduced to

$$
\begin{align*}
& X=-\int_{c}\left[A-\frac{1}{2} \rho\left(u^{2}+v^{2}\right)\right] d y=\frac{1}{2} \rho \int_{c}\left(u^{2}+v^{2}\right) d y  \tag{13}\\
& Y=\int_{c}\left[A-\frac{1}{2} \rho\left(u^{2}+v^{2}\right)\right] d x=-\frac{1}{2} \rho \int_{c}\left(u^{2}+v^{2}\right) d x,  \tag{14}\\
& \text { and } \quad M=\int_{c}\left[A-\frac{1}{2} \rho\left(u^{2}+v^{2}\right)\right](x d x+y d y)=-\frac{1}{2} \rho \int_{c}\left(u^{2}+v^{2}\right)(x d x+y d y) \tag{15}
\end{align*}
$$

while simplifying (13),(14), and (15), we have to use the following results

$$
\int_{c} d y=\int_{c} d y=\int_{c} x d x=\int_{c} x d y=0
$$

Which hold good because $C$ is closed contour.
Now the contour of the cylinder is a streamline. Hence we have $d x / u=d y / v$. Now,

$$
\begin{gather*}
\frac{d x}{u}=\frac{d y}{v}=\frac{d x+\iota d y}{u+i v}=\frac{d x-\iota d y}{u-\iota v} \text { or } \frac{d x-\iota y}{d x+\iota d y}=\frac{u-\iota v}{u+\iota v}=\frac{(u-\iota v)^{2}}{u^{2}+v^{2}} \\
(u-\iota v)^{2}(d x+\iota d y)=\left(u^{2}+v^{2}\right)\left(d x+\frac{1}{\iota} d y\right) \tag{16}
\end{gather*}
$$

from (13) and (14) we have

$$
\begin{aligned}
X-\iota Y & =\frac{1}{2} \rho \int_{c}\left(u^{2}+v^{2}\right)(d y+\iota d x)=\frac{1}{2} \rho \iota \int_{c}\left(u^{2}+V^{2}\right)\left(d x+\frac{1}{\iota} d y\right) \\
& =\frac{1}{2} \int_{c}\left(u^{2}+v^{2}\right)(d x-\iota d y)=\frac{1}{2} \rho \iota \int_{c}(u-\iota v)^{2}(d x+\iota y), \quad \text { by } \\
& =\frac{1}{2} \rho \iota \int_{c}\left(\frac{d w}{d z}\right)^{2} d z, \quad \operatorname{using}(12) \text { and the fact } z=x+\iota y \Rightarrow d z=d x+\iota d y
\end{aligned}
$$

$$
\begin{aligned}
M & =\text { Real part of }-\frac{1}{2} \rho \int_{c}(x+\iota y)(d x-\iota d y)\left(u^{2}+v^{2}\right) \\
& =\text { Real part of }-\frac{1}{2} \rho \int_{c}(x+\iota y)(u-\iota v)^{2}(d x+\iota d y), \quad \operatorname{using}(16) \\
& =\text { Real part of }\left\{-\frac{1}{2} \rho \int_{c} z\left(\frac{d w}{d z}\right)^{2} d z\right\}
\end{aligned}
$$

Remark 1. The above integrals are to be taken over the contour of the cylinder. If however, we hake a large contour surrounding the cylinder such that between this contour and the cylinder there is no singularity of the integrand, then we can take the integrals round such large contours. The singularities of the integrand occur at sources, sinks, sublets etc.

Remark 2. In what follows, we shall often use the following important definitions and results of functions of complex variables.

A point at which a function $f(z)$ ceases to be analytic is known as a singular point or singularity of the function. If in the neighbourhood of the point $z=a, f(z)$ can be expanded in positive and negative power of $(z-a)$, say

$$
f(z)=\cdots+A_{2}(z-a)^{2}+A_{1}(z-a)+A_{0}+\frac{B_{1}}{z-a}+\frac{B_{2}}{(z-a)^{2}}+\ldots
$$

then the point $z=a$ is a singular point of $f(z)$. If only a finite number of terms contain negative powers of $z-a$, the point $z=a$ is called a pole. In this case the coefficient of $1 /(z-a)$ is called the residue fo the function at $z=a$.

### 3.1 Cauchy's Residue theorem

If $f(z)$ is analytic, except at a finite number of poles within a closed contour $C$ and continuous on the boundary $C$, then
$\int_{c} f(z) d z=2 \pi \iota \times$ [sum of the residues of $f(z)$ at its poles within $C$ ]
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