Simply the circle theorem(14)

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1 The Milne-Thomson circle theorem or simply the circle theorem

Statement 1. *Let $f(z)$ be the complex potential for a flow having no rigid boundary and such that there are no singularities of flow within the circle $|z| = a$. Then, on introducing the solid circular cylinder $|z| = a$ into the flow, the new complex potential is given by $w = f(z) + \overline{f}(a^2/z)$ for $|z| \geq a$.*

Proof. Let C be the cross-section of the circular cylinder $|z| = a$. Then on $Ck, z\overline{z} = a^2$ or $\overline{z} = a^2/z$.Hence for points on the circle, we have

$$w = f(z) + \overline{f}(a^2/z) = f(z) + \overline{z}(\overline{z}) \quad \text{or} \quad \phi + \iota\psi = f(z) + \overline{f}(\overline{z}) \tag{1}$$

Since the quantity on R.H.S of 1 is purely real, equating imaginary parts 1 gives $\psi = 0$ on C. Hence C is a streamline in the new flow.

By hypothesis all the singularities of $f(z)$ (at which sources, sinks, doublets or vertices may be present) lie outside the circle $|z| = a$ and so the singularities of $f(a^2/z)$ lie inside the circle $|z| = a$. Hence the singularities of $\overline{f}(a^2/z)$ also lie inside the circle $|z| = a$. Thus we find the additional term $\overline{f}(a^2/z)$ introduces no new singularities into the flow outside the circle $|z| = a$.

Hence $|z| = a$ is a possible boundary for the new flow and $w = f(z) + \overline{f}(a^2/z)$ is the appropriate complex potential for the new flow.

Remark 1. *In the above proof of circle theorem we have used the following important results ;*

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Let $u(t)$ and $v(t)$ be real functions of a real variable t. Let $f(t) = u(t) + \iota v(t)$ so that $f(t)$ is a complex function of the real variable t. Then conjugate of $f(t)$ is denoted and defined as

$$\overline{f}(t) = u(t) - \iota v(t)$$

On replacing real variable t by the complex variable $z = (x + \iota y)$, $f(z)$ and $\overline{f}(z)$ are defined as follows :

$$f(z) = u(z) + \iota v(z), \qquad \overline{f}(z) = u(z) - \iota v(z)$$
$$Again, \quad f(\overline{z}) = u(\overline{z}) + \iota v(\overline{z}), \qquad \overline{f} = u(\overline{z}) - \iota v(\overline{z})$$

On Comparing the forms of $f(z)$ and $f(\overline{z})$, we find that, since $z = x - \iota y$, the value of $\overline{f}(\overline{z})$ is obtained from $f(z)$ by replacing ι throughout by $-\iota$. It then follows that $\overline{f}(\overline{z})$ is merely the complex conjugate of $f(z)$ and accordingly, we write $\overline{f}(\overline{z}) = \overline{f(z)}$.

Remark 2. *When a circular cylinder is present in the field of sources, sinks, doublets of vortices, the above theorem provides an easy method for determining the image system. Furthermore the theorem can also be used to determine modified flows when a long circular cylinder is introduced into a given two-dimension flow. Consider the following application of "circle theorem"*

□

2 To determine image system for a source outside a circle (or a circular cylinder) of radius a with help of the circle theorem.

In the figure. Let $OA = f$. Suppose there is a source of strength m at A where $z = f$, outside the circle of radius a whose center is at O. When the source is alone in the fluid the complex potential at a point $P(z)$ is given by

$$f(z) = -m \log(z - f) \quad Then \quad \overline{f}(z) = -m \log(z - f)$$
$$\overline{f}(a^2/z) = -m \log\big(a^2/z - f\big)$$

When the circle of section $|z| = a$ is introduced, then the complex potential in the region $|z| \geq a$ is given by

$$\begin{aligned} w &= f(z) + \overline{z}(a^2/z) = -m \log(z - f) - m \log\big(a^2/z - f\big) \\ &= -m \log(z - f) - m \log\left(\frac{a^2 - zf}{z}\right) \\ &= -m \log(z - f) - m \log\big(a^2 - zf\big) + m \log z \\ &= -m \log(z - f) - m \log[(-f)(z - a^2/f)] + m \log z \\ &= -m \log(z - f) - m \log[(-f)(z - a^2/f)] + m \log z - m \log(-f) \\ \therefore \quad w &= -\log(z - f) - m \log\big(z - a^2/f\big) + m \log z + constant, \end{aligned} \tag{2}$$

the constant (real or complex, $-m\log(-f)$) being immaterial from the view point of analysing the flow.(2) shows that w is the complex potential of

1. a source m at $A, z = f$
2. a source m at $B, Z = a^2/f$
3. a sink $-m$ at the origin

Since $OA \ldots OB = a^2$, A and B are the inverse points with respect to the circle $|z| = a$ and so B is inside the circle.

Thus the image system for a source outside a circle consist of an equal source at the inverse point and an equal sink at the center of the circle.

All the best...
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