Exactly Soluble System (II)
(d)

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THE ASSOCIATED LAGUERRE EQUATION

Solutions to the associated Laguerre differential equation with $\nu \neq 0$ and k an integer are called associated Laguerre Polynomials. Associated Laguerre Polynomials are implemented in the Wolfram Language as LaguerreL(n, k, x) in terms of the unassociated Laguerre Polynomials. The associated Polynomials solve a related set of equations given by differentiating the Laguerre equation for L_{n+k}, k times.

$$0 = \frac{d^k}{dx^k}\left(x\frac{d^2L_{n+k}}{dx^2} + (1-x)\frac{dL_{n+k}}{dx} + n+k\,L_{n+k}\right)$$

$$= \frac{d^k}{dx^k}\left(x\frac{d^2L_{n+k}}{dx^2}\right) + \frac{d^k}{dx^k}\left((1-x)\frac{dL_{n+k}}{dx}\right) + n+k\,\frac{d^kL_{n+k}}{dx^k}$$

For the first term –

$$\frac{d}{dx}\left(x\frac{d^2L_{n+k}}{dx^2}\right) = x\frac{d^3L_{n+k}}{dx^3} + \frac{d^2L_{n+k}}{dx^2}$$

$$\frac{d^2}{dx^2}\left(x\frac{d^2L_{n+k}}{dx^2}\right) = x\frac{d^4L_{n+k}}{dx^4} + 2\frac{d^3L_{n+k}}{dx^3}$$

$$\frac{d^3}{dx^3}\left(x\frac{d^2L_{n+k}}{dx^2}\right) = x\cdot\frac{d^5L_{n+k}}{dx^5} + 3\frac{d^4L_{n+k}}{dx^4}$$

The Pattern is emerging

$$\frac{d^k}{dx^k}\left(x \cdot \frac{d^2 L_{n+k}}{dx^2}\right) = x\frac{d^{k+2} L_{n+k}}{dx^{k+2}} + k \cdot \frac{d^{k+1} L_{n+k}}{dx^{k+1}}$$

check one more derivative to complete the induction.

$$\frac{d^{k+1}}{dx^{k+1}}\left(x \frac{d^2 L_{n+k}}{dx^2}\right) = \frac{d}{dx}\left(x \frac{d^{(k+1)+2} L_{n+k}}{dx^{(k+1)+2}} + (k+1)\frac{d^{(k+1)+1} L_{n+k}}{dx^{(k+1)+1}}\right.$$

So the form is correct.

For the second term, we need

$$\frac{d^k}{dx^k}(1-x)\frac{dL_{n+k}}{dx} = \frac{d^k}{dx^k}\left(\frac{dL_{n+k}}{dx} - x\frac{dL_{n+k}}{dx}\right)$$

$$= \frac{d^{k+1} L_{n+k}}{dx^{k+1}} - \frac{d^k}{dx^k}\left(x \frac{dL_{n+k}}{dx}\right.$$

the last part -

$$\frac{d}{dx}\left(x \cdot \frac{dL_{n+k}}{dx}\right) = x \cdot \frac{d^2 L_{n+k}}{dx^2} + \frac{dL_{n+k}}{dx}$$

$$\frac{d^2}{dx^2}\left(x \cdot \frac{dL_{n+k}}{dx}\right) = x \cdot \frac{d^3 L_{n+k}}{dx^3} + 2 \cdot \frac{d^2 L_{n+k}}{dx^2}$$

So the generic term is

$$\frac{d^k}{dx^k}\left(x \frac{dL_{x+k}}{dx}\right) = x\frac{d^{k+1} L_{n+k}}{dx^{n+k}} + k\frac{d^k L_{n+k}}{dx^k}$$

and check one more

$$\frac{d^{k+1}}{dx^{k+1}}\left(x \frac{dL_{n+k}}{dx}\right) = \frac{d^{k+1} L_{n+k}}{dx^{k+1}} + x\frac{d^{k+2} L_{n+k}}{dx^{k+2}} + k\frac{d^{k+1} L_{n+k}}{dx^{k+1}}$$

$$= x\frac{d^{(k+1)+1} L_{n+k}}{dx^{(k+1)+1}} + (k+1)\frac{d^{k+1} L_{n+k}}{dx^{k+1}}$$

Therefore, returning to the equation.

$$0 = \frac{d^K}{dx^K}\left(x \frac{d^2 L_{n+K}}{dx^2}\right) + \frac{d^K}{dx^K}\left[(1-x)\frac{dL_{n+K}}{dx} + (n+K)\frac{d^K L_{n+K}}{dx^K}\right]$$

$$= x\frac{d^{K+2}L_{n+K}}{dx^{K+2}} + K\frac{d^{K+1}L_{n+K}}{dx^{K+1}} + \frac{d^{K+1}L_{n+K}}{dx^{K+1}} - x\frac{d^{K+1}L_{n+K}}{dx^{K+1}} - K\frac{d^K L_{n+K}}{dx^K} + (n+K)\frac{d^K L_{n+K}}{dx^K}$$

$$= x\frac{d^2}{dx^2}\left(\frac{d^K L_{n+K}}{dx^K}\right) + (K+1-x)\frac{d}{dx}\left(\frac{d^K L_{n+K}}{dx^K}\right) + n\left(\frac{d^K L_{n+K}}{dx^K}\right).$$

and, inserting a minus sign, the associated Laguerre equation is

$$x \cdot \frac{d^2 L_n^K}{dx^2} + (K+1-x)\frac{dL_n^K}{dx} + nL_n^K = 0$$